

# ENDOGENOUS PARTY PLATFORMS: “STOCHASTIC” MEMBERSHIP

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ABSTRACT. We analyze existence of divergent equilibrium in a model of endogenous party platforms with stochastic parties. The parties proposals depend on their membership, while the membership depends both on the proposals and the unobserved idiosyncratic preferences of citizens over parties. It is shown that when citizens view the parties as similar, apart from their policy proposals (i.e., the party platform is a good predictor of individual membership decision), the divergent equilibria exist.

## 1. INTRODUCTION

The issue of party platform formation has been a subject of substantial attention in political economy. The major idea in this literature is that platforms of political parties are formed in response to preferences of their members, whereas the memberships themselves are, at least in part, determined by the platforms. Thus, in equilibrium the party platforms should respond to the preferences of members that they attract. An early paper putting forward a political competition framework to define an equilibrium concept in which party ideology and its membership are endogenously determined was Baron (1993). This equilibrium concept was related to the one used in the “voting with one’s feet” models developed in the study of local public goods (see Caplin and Nalebuff 1997 for an abstract framework that covers both the political economy and public finance applications). In related work Aldrich (1983a,b), Gerber and Ortuño-Ortín (1998), and Poutvaara (2003) have considered the interrelationship between partisan policy platforms and political activism.

A major issue in this literature has been the problem of existence of *divergent* equilibria, in which parties, though, possibly, *ex ante* identical propose different policies and attract members with different policy preferences. In a deterministic model of this type (such as Ortuño-Ortín and Roemer1998 or Gomberg, Marhuenda and Ortuño-Ortín 2004) such an equilibrium, if it exists, involves a full sorting of agents in terms of their preferences over the policy space: even minute policy differences between parties induce a unique party choice by almost all citizens (in the party activist literature, along the lines of Aldrich (1983a), where there is a third possibility - that of non-participation - is available, it is still normally assumed

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preliminary and incomplete.

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that those actually actively taking part in partisan activities do it in the ideologically closest party). However, such perfect sorting is not commonly observed in reality: even ideologically identical people may frequently find themselves in different parties based on idiosyncratic non-policy considerations (perhaps, historical esthetical or personal). These non-policy issues might not even be observable by an outsider, making the observed policy preferences only stochastic predictors of individual party choice. This is, of course, not a new idea in political science, where the study of stochastic models of voting have been widespread for a long time (see Coughlin 1992 for a survey). Our “stochastic” model of endogenous membership follows some of the same intuitions. Our focus is, however, somewhat different. In particular, rather than considering the vote-maximizing parties in an electoral context we restrict our attention to parties aggregating members’ preferences and try to establish to which extent the results of an older deterministic models (such as our own Gomberg *et al.* 2004) extend to this new setting. In modeling parties as aggregating preferences of their members, while membership is, in turn, determined in part (but not fully) by party policy positions our paper is related to a recent work by Roemer (2007). Our approach, however, is different in such crucial aspects as, among others, our more explicit modeling of membership decisions, the nature of intraparty decision rules (which in Roemer’s case discriminate among members of different ideologies based on belonging to a “partisan core”). Our objective is likewise distinct: we want to establish to which extent the results for the stochastic membership model may be viewed as an extension of those for the older deterministic model.

A seemingly major difficulty in this extension is that the studies of the deterministic model have crucially used the sorting nature of equilibrium to derive the results from the properties of the space of sorting partitions (see Caplin and Nalebuff 1997 and Gomberg *et al.* 2004; in the context of local public goods this approach goes back to Westhoff 1977). In the absence of perfect this approach is, of course, not feasible. However, the crucial feature of the deterministic model is, in fact, not the sorting *per se*, but the instability of pooling: if two, in other respects identical, parties propose the same policy than the entire population is indifferent and can be split to support that as an equilibrium, but even minor policy perturbations would result in full population sorting and sharply divergent policies. Thus, whenever it is possible to show that such an equilibrium may not be unique, existence of sorting equilibrium is, in fact, guaranteed! In this paper we show that this intuition, in part, extends to the stochastic context: if the convergent equilibrium exists but is unstable to small policy perturbations, it may be used to detect existence of divergent equilibria. In fact, as the addition of the stochastic component adds continuity to the model, in a sense the results become, in fact, more transparent in this setting. In particular, in a context of a “generalized example” in our framework, we show that when parties are perceived by voters to be very similar in non-policy terms, so that the observed randomness of individual partisan choice is relatively small, the results of the deterministic model extend to the stochastic case.

For the moment (and for simplicity) we abstract from possible voting game participation by parties (in the terminology of Caplin and Nalebuff 1997 our parties are “membership-based”). The main reason here is methodological: we believe that the issue of endogenizing party membership is distinct from the issue of strategic behavior by party leaders in a democratic election. Our main concern here is the

former, and we want to consider it separately. This assumption may be viewed as appropriate for either a model of parties in a setting without commitment (*e.g.*, when voters would not believe a party, once in office, can implement policies not supported by its membership) or in a setting without true electoral competition (*e.g.*, if parties' share of the office is determined through non-electoral means). Of course, we do intend to explore extending our results to cover the case of possible strategic interactions between parties.

The rest of this paper is organized as follows. Section 2 presents the model and develops a general existence result, section 3 presents the results for the mean and median voter rules in a single dimension of issue space and section 4 concludes.

## 2. MODEL

The model in this section is closely related Gomberg *et al.* (2004), simplified to assume sincere voting by agents and extended to cover an unobservable (by outsiders) dimension, in which the voters rank parties according to their intrinsic features. There are  $m = 2$  parties, that propose policy vectors  $x_j \in X$ , where  $X$  is a non-empty compact and convex subset of  $\mathbb{R}^n$ . In addition to a policy  $x_j$  a party is characterized by a non-policy variable  $y \in \mathbb{R}$ , which may be interpreted as reflecting currently fixed non-political characteristics, which may matter for individual preferences.

There is a continuum of agent types with preferences over both policy and non-policy characteristics of parties. Specifically, each agent of type  $(\alpha, \varepsilon) \in A \times E \subset \mathbb{R}^n \times \mathbb{R}$  has Euclidean preferences represented by the utility function

$$u(x, y; \alpha) = -\|(x, y) - (\alpha, \varepsilon)\|$$

where  $x \in X, y \in Y$  are, respectively, the policy platform and intrinsic characteristic of the party s/he joins. We shall take  $A = X$ , so that (for a fixed  $y$ ) an agent may be identified with his/her ideal policy.

There is a measure space of agents (citizens)  $(A \times E, \mathcal{B}, \nu)$  such that  $\nu(A \times E) = 1$ . We denote the distribution function of  $\nu$  as  $F(\alpha, \varepsilon)$  and we assume existence of a non-zero continuous density function  $f(\alpha, \varepsilon)$  with conditional density functions denoted as  $f_1(\alpha|\varepsilon)$  and  $f_2(\varepsilon|\alpha)$ . We shall assume that citizens join parties they like the most. While the individual party choice is, thus, unambiguous, in this paper we shall consider the case when the second coordinate of individual type  $\varepsilon$  is unobservable by others (it may express idiosyncratic individual attitudes towards parties). Thus, from the point of view of both outside observers and of parties themselves the observable individual preferences over policies (given by  $\alpha$ ) may serve only as an imperfect predictor of individual party choice: the latter appears stochastic.

Ignoring zero-measure sets, a party membership is observed as a population measure  $\sigma_j$  on  $A$ . We shall denote the corresponding population density as  $g_j(\alpha)$ . A population partition  $\sigma = (\sigma_1, \sigma_2)$  shall be considered *admissible* if for every  $S \in \mathcal{B}$ , we have that  $\sigma_1(S) + \sigma_2(S) = \nu(S)$ . The set of admissible population partitions shall be denoted as  $\Sigma$ . On the space  $\Sigma$  we consider the supremum norm, so that  $(\Sigma, \|\cdot\|)$  is a Banach space. On  $\Sigma^2$  we consider the product topology.

A political party  $j = 1, 2$  chooses its policy by aggregating the observed policy preferences of its members according to some fixed rule  $P_j$ , which we shall call its statute. As parties do not observe  $\varepsilon$  the aggregation applies only to  $\alpha$ . As an

example of such rule we may consider, for instance the mean voter rule

$$P_j(g_j) = \frac{\int \alpha g_j(\alpha) d\alpha}{\int g_j(\alpha) d\alpha}$$

which assigns to each party an ideal policy of its mean voter. We shall denote the profile of party statutes as  $P = (P_1, P_2)$ . The mapping  $P : \Sigma^2 \rightarrow X^2$  assigns to every population partition a policy profile  $x = (x_1, x_2)$ .

Given the exogenous idiosyncratic party characteristic parameters  $y_1$  and  $y_2$  and given a policy proposal profile  $x = (x_1, x_2) \in X^2$  we can define  $\sigma(x) = (\sigma_1(x), \sigma_2(x))$  to be the partition given by the party density functions

$$g_j^x(\alpha) = \int_{\{(\alpha, \varepsilon) : \|(x_j, y_j) - (\alpha, \varepsilon)\| \geq \|(x_i, y_i) - (\alpha, \varepsilon)\|, i \neq j\}} f(\alpha, \varepsilon) d\varepsilon$$

This is the population partition that results from citizens joining the parties that give them highest utility. Without loss of generality, we may assume from now that  $y_2 > y_1$ .

Hence, when parties choose the proposals  $x = (x_1, x_2)$ , the agents affiliate to one or the other party, depending on which side of the line

$$z^x(t) = \frac{x_1 - x_2}{y_2 - y_1}t + \frac{y_1 + y_2}{2} - \frac{x_1^2 - x_2^2}{2(y_2 - y_1)}$$

they fall. For example,

$$\begin{aligned} g_1^x(\alpha) &= \int_{-\infty}^{z^x(\alpha)} f(\alpha, \varepsilon) d\varepsilon \\ g_2^x(\alpha) &= \int_{z^x(\alpha)}^{+\infty} f(\alpha, \varepsilon) d\varepsilon \end{aligned}$$

**Lemma 1.** The mapping  $\sigma_i : X^2 \rightarrow \Sigma$  is Fréchet differentiable<sup>1</sup>, for each  $i = 1, 2$ .

**Proof:** We do the proof for  $i = 1$ . Let  $x = (x_1, x_2) \in X^2$ . Note that

$$\frac{d}{dh} \Big|_{h=0} g_1^{(x_1+ht, x_2)}(\alpha) = \partial_{x_1} g_1^x(\alpha) = f(\alpha, z^x(\alpha)) \partial_h z^{(x_1+ht, x_2)}(\alpha) = f(\alpha, z^x(\alpha)) \frac{\alpha - x_1}{y_2 - y_1} t$$

Likewise, one computes

$$\frac{d}{dh} \Big|_{h=0} g_1^{(x_1, x_2+hs)}(\alpha) = f(\alpha, z^x(\alpha)) \frac{x_2 - \alpha}{y_2 - y_1} s$$

From the above expressions, the Gateaux derivative can be easily computed. Now let  $\delta > 0$ . By Lebesgue's Theorem, we have

$$\begin{aligned} \int_{z^x(\alpha)}^{z^{(x_1+t, x_2+s)}(\alpha)} f(\alpha, \varepsilon) d\varepsilon &\leq (z^{(x_1+t, x_2+s)} - z^x(\alpha)) (f(\alpha, z^x(\alpha)) + \delta) = \\ &= \left( \frac{\alpha(s-t) + x_2s - x_1t}{y_2 - y_1} + \frac{t^2 - s^2}{2(y_2 - y_1)} \right) (f(\alpha, z^x(\alpha)) + \delta) \end{aligned}$$

<sup>1</sup>see Luenberger (1969) for definitions of differentiability of functions from and into function spaces.

whenever  $|t|$  and  $|s|$  are small enough. Thus,

$$\begin{aligned} g_1^{(x_1+t, x_2+s)}(\alpha) - g_1^{(x_1, x_2)}(\alpha) - f(\alpha, z^x(\alpha)) \frac{\alpha - x_1}{y_2 - y_1} t - f(\alpha, z^x(\alpha)) \frac{x_2 - \alpha}{y_2 - y_1} s &= \\ &= \int_{z^x(\alpha)}^{z^{(x_1+t, x_2+s)}(\alpha)} f(\alpha, \varepsilon) d\varepsilon - f(\alpha, z^x(\alpha)) \frac{\alpha(s-t) + x_2 s - x_1 t}{y_2 - y_1} = \\ &= \left( \frac{t^2 - s^2}{2(y_2 - y_1)} \right) (f(\alpha, z^x(\alpha)) + \delta) \end{aligned}$$

Hence, given  $\delta > 0$ , if  $|t|$  and  $|s|$  are small enough we have that

$$\begin{aligned} \sup_{\alpha \in A} \left( g_1^{(x_1+t, x_2+s)}(\alpha) - g_1^{(x_1, x_2)}(\alpha) - f(\alpha, z^x(\alpha)) \frac{\alpha - x_1}{y_2 - y_1} t - f(\alpha, z^x(\alpha)) \frac{x_2 - \alpha}{y_2 - y_1} s \right) \\ \leq \left( \frac{t^2 - s^2}{2(y_2 - y_1)} \right) (\|f\| + \delta) \end{aligned}$$

Since,  $\delta > 0$  is arbitrary, we see that

$$\lim_{(t, s) \rightarrow 0} \frac{\left\| g_1^{(x_1+t, x_2+s)}(\alpha) - g_1^{(x_1, x_2)}(\alpha) - f(\alpha, z^x(\alpha)) \frac{\alpha - x_1}{y_2 - y_1} t - f(\alpha, z^x(\alpha)) \frac{x_2 - \alpha}{y_2 - y_1} s \right\|}{\|(t, s)\|} = 0$$

and the Lemma is proved.

The multiparty equilibrium under free mobility of population may be defined as follows:

**Definition 2.** Given the profile of party statutes  $P$  and the idiosyncratic party characteristics  $y_1, y_2$  we say that  $(x^*, \sigma^*) \in X^m \times \Sigma$  is a **multi-party equilibrium** if:

- (i)  $x^* = P(\sigma^*)$
- (ii)  $\sigma^* = \sigma(x^*)$

Furthermore, the equilibrium is **divergent** if  $x_1^* \neq x_2^*$ .

Consider the mapping  $\phi : X^2 \rightarrow X^2$  defined by

$$\phi(x) = P(\sigma(x))$$

Clearly, an equilibrium is just a fixed point of this mapping.

Note that if  $y_1 = y_2$ , so that individual membership is fully determined by policy positions, this model becomes deterministic and fully falls into the framework posed by Caplin and Nalebuff (1997). Thus, the interesting case for us in this paper is  $y_1 \neq y_2$ . In fact, as long as what we are interested in is simply existence of *some* multi-party equilibrium, the present model still fits the same approach: imposing basic continuity and minimal internal support assumptions on the statutes  $P_j$  (as in Gomberg *et al.* 2004), together with the exogenously imposed difference between parties, would, indeed, lead into existence of an equilibrium, which, in fact, would involve full sorting of agents in the  $A \times E$  space. This sorting, however, would be incomplete once projected onto the observable  $A$  space. In fact, there is nothing that prevents the observed policy positions  $x_j$  of the parties to coincide (converge). Indeed, whenever such equilibrium exists, the older results, by themselves, are silent on the existence of divergent equilibria. It is establishing conditions for existence of this last equilibrium type that concerns us in this work.

Suppose the two parties are *ex ante* identical (i.e., they use the same policy rule  $P_1 = P_2$ ) and the distribution of  $\alpha$  is independent of  $\varepsilon$ , so that  $f_1(\alpha|\varepsilon) = f_1(\alpha)$ . In this case, if  $x_1 = x_2$

$$g_1(\alpha) = \int_{\alpha \leq \frac{y_1+y_2}{2}} f(\alpha, \varepsilon) d\varepsilon = \int_{\alpha \geq \frac{y_1+y_2}{2}} f(\alpha, \varepsilon) d\varepsilon = g_2(\alpha)$$

Then, clearly, the convergent equilibrium shall exist. Our question in this paper shall be to find out the conditions for existence of other, divergent, equilibria.

In order to do this, we shall, following Caplin and Nalebuff (1997) and Gomberg *et. al* (2004), impose a number of assumptions on policy rules. The two previously introduced assumptions are continuity and minimal internal support (the latter ensures that parties would choose interior ideological positions).

**Assumption 1** (*continuity*)  $P_j : (\Sigma, \|\cdot\|) \rightarrow X$  is continuous and Fréchet differentiable, for  $j = 1, 2$

**Assumption 2** (*minimal internal support*) For each party  $j$  and any value of  $y$  there exists a  $\delta > 0$  such that for any population partition  $\sigma$  and any  $x_j \in X$

$$\frac{1}{\sigma_j(A)} \int_{\{\alpha: u(P_j(\sigma), y; \alpha) > u(x_j, y; \alpha)\}} g_j(\alpha) d\alpha > \delta$$

Strictly speaking, if all we care is existence of *some* equilibrium, with population sorted in  $A \times E$  (in the sense of Caplin and Nalebuff 1997), the above two assumptions are sufficient, as long as  $y_1 \neq y_2$ . The proof of this fact could be done by using the techniques of Caplin and Nalebuff (1997) and our earlier work, which considered the fixed point mapping  $\psi : \Sigma \rightarrow \Sigma$ , where the population partitions that could emerge were described by hyperplanes. If we, furthermore, take into account that as the party idiosyncratic characteristics are distinct the population partitions cannot be “reversed” (in fact, if a certain population partition  $\sigma \in \Sigma$  may be induced by some pair of  $x$  and  $y$  it’s negative  $-\sigma$  can’t), we shall observe that the equilibrium must exist for any dimension of the policy space (as the space of population partitions in the image of  $\psi$  is, in this case, contractible). However, there is nothing that guarantees that such equilibrium would result in distinct policy positions by parties: the sorting may be entirely caused by the difference in  $y$ ’s.

In a deterministic version of the model Caplin and Nalebuff (1997) have postulated the assumption that the party policy rules would never result in identical policies if party populations have opposing preference, in the sense of being divided by a hyperplane in the ideological space. This assumption, problematic even in that model, unless the policy rules are just aggregating intraparty preferences (“membership-based” in their terminology) would be entirely unapplicable here, as, in general, the sorting is not perfect in this model. Fortunately, it turns out, that what was driving the Caplin and Nalebuff result was not this, but a weaker condition: instability of the convergent equilibrium under adjustment dynamics. Denoting  $\Delta = \{(x, x') \in X^2 : x = x'\}$ , the diagonal of the policy space, we shall assume the following:

**Assumption 3** (*instability of pooling*). Let  $x^* \in \Delta \subset X^2$  be an equilibrium policy profile. Then, there exists an open neighborhood  $O \subset X^2$  containing  $x^*$  such that for any boundary point  $x \in \partial O \setminus \Delta$ , we have that  $\phi(x) \notin O$ .

Intuitively, this states that once the convergent equilibrium is perturbed, the induced population partition induces a further policy divergence. It is not difficult to check that in a deterministic model the requirement of distinct policies from sorting partitions imposed in earlier work implies this (any minute policy difference in a deterministic model induces a full population sorting which, in turn, induces distinct policies, which, by continuity of policy rules, cannot be close to the diagonal). This weaker assumption is, in fact, sufficient for our first result.

**Remark 3.** Note that if the proposal of the parties satisfy  $x = (x_1, x_2) \in \Delta$ , then the induced partition of the population is described by the density functions  $\frac{1}{\lambda_1}g_1^x(\alpha) = \frac{1}{\lambda_2}g_2^x(\alpha) = f_1(\alpha)$ , where  $\lambda_i$  is simply the vote share of party  $i = 1, 2$  that results from this partition ( $\lambda_1 + \lambda_2 = 1$ ). It should be noted that the vote shares are themselves independent of the common ideological position of the parties and is entirely determined by  $y_1$  and  $y_2$ .

In particular, for  $i = 1, 2$ ,  $\sigma_i$  is constant on the diagonal  $\Delta \subset X^2$ . It follows that there can be at most a unique pooling equilibrium policy profile  $x^*$ .

**Proposition 4.** Let Assumptions 1, 2 and 3 hold. Then, there is an equilibrium  $x^*$ .

**(Proof):** Consider the mapping  $\phi : X^2 \rightarrow X^2$  defined by  $\phi(x) = P(\sigma(x))$ . The fixed points of this mapping correspond to equilibria of the model. The mapping is clearly defined on the entire  $X^2$  and continuity follows from assumption 1. As  $X^2$  is compact and convex, by Brouwer's fixed point theorem there must exist at least one such fixed point (possibly convergent).

**Proposition 5.** Let  $P_1 = P_2$  and the distributions of  $\alpha$  and  $\varepsilon$  are independent. Then, there is unique convergent equilibrium  $x^* \in \Delta$ .

**(Proof):** If the distributions of  $\alpha$  and  $\varepsilon$  are independent and  $x \in \Delta$ , then the population partition is entirely determined by the  $y$ 's and the induced population measures on  $A$  must coincide ( $g_1^x(\alpha) = g_2^x(\alpha) = f_1(\alpha|\varepsilon) = g^*(\alpha)$  is independent of  $\varepsilon$  and  $x$ ). Therefore, there must exist a unique convergent equilibrium policy profile  $x^*$  such that  $x_1^* = x_2^* = P_j(g^*(\alpha))$ .

**Proposition 6.** Let  $P_1 = P_2$  and the distributions of  $\alpha$  and  $\varepsilon$  are independent. Suppose the unique convergent equilibrium  $x^* \in \Delta$  is a Lefschetz fixed point of  $\phi^2$ . If the dimension of the policy space  $n$  is odd and assumptions 1-3 hold then, there shall exist a divergent equilibrium.

**(Proof):** The equilibrium  $x^* \in \Delta$  is *stable* along the diagonal,  $\Delta$ , since any policy profile on  $\Delta$  is mapped directly into  $x^*$ . Furthermore, this equilibrium is either isolated or else, there is at least a divergent equilibrium. Thus, we may assume this equilibrium is an isolated fixed point.

We will show that  $x^*$  cannot be the unique equilibrium. Assume, for contradiction that  $x^*$  is the unique equilibrium. We argue that the boundary of  $X^2$  is unstable. Let  $x = (x_1, x_2) \in \partial X^2$ . By making  $x'_j$  arbitrarily close to  $x_j$ , and using assumption 2, we can make an arbitrarily large proportion of party members strictly prefer  $x'_j$ . Hence,  $\phi(x) \in \text{int}(X^2)$  and the boundary of  $X^2$  is unstable.

As  $\phi$  is a mapping from the compact and convex set  $X^2$  to itself, and as  $x^*$  is assumed to be Lefschetz, by the Lefschetz Fixed Point Theorem (see Guillemin and

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<sup>2</sup>That is, if all the eigenvalues of  $d_x\phi$  are all unequal to +1.

Pollack 1974, pp. 119-130) the total sum of the indices of the fixed points  $x^*$  must be equal to 1 (the Euler characteristic of  $X^2$ ). Recall, that the index  $\text{ind}(x^*)$  of a Lefschetz fixed point  $x^*$  may be calculated as  $(-1)^d$ , where  $d$  is the dimension of the unstable manifold of  $x^*$ .

Now, by assumption 3, the equilibrium  $x^*$  is unstable off diagonal. As the co-dimension of the diagonal  $\Delta$  is  $n$ , the index of the diagonal fixed point equals  $(-1)^n$ , which implies it cannot be unique if  $n$  is odd. Hence, a divergent equilibrium must exist.

Of course, this proposition, on its own, is of limited interest: unless we can show that assumption 3 holds for cases where  $y_1 \neq y_2$  the result is vacuous. Fortunately, for a restricted case when the parties use common preference aggregation rules, such as the mean and the median voter rule, we can show that for the difference between parties sufficiently small the assumption does hold.

### 3. THE MEAN AND MEDIAN VOTER RULES

Consider, first, the mean voter rule defined above, *i.e.* suppose that for any  $g_j$  each party chooses

$$P_j(g_j) = \frac{\int_A \alpha g_j(\alpha) d\alpha}{\int_A g_j(\alpha) d\alpha} = \frac{\int_A \alpha g_j(\alpha) d\alpha}{\sigma_j(A)}$$

It turns out that as the  $y$ 's get closer (but remain distinct), existence of equilibrium can be assured. In fact, the following proposition holds:

**Proposition 7.** Let  $n = 1$  and the parties be using the mean voter rule. For any overall population distribution  $\nu$  on  $A \times E$  such that  $\alpha$  and  $\varepsilon$  are independent and a continuous density function  $f(\alpha, \varepsilon)$  exists. If

$$\frac{1}{\lambda_1(1-\lambda_1)} \int_A (\alpha - \mu)^2 f(\alpha, \frac{y_1 + y_2}{2}) d\alpha > |y_2 - y_1| > 0$$

then, there exists a divergent equilibrium.

**Proof of Proposition 7:** Recall that, given the proposals  $x = (x_1, x_2)$  of the parties, we use the notation

$$g_j^x(\alpha) = \int_{\{(\alpha, \varepsilon): \|(x_j, y_j) - (\alpha, \varepsilon)\| \geq \|(x_i, y_i) - (\alpha, \varepsilon)\|, i \neq j\}} f(\alpha, \varepsilon) d\varepsilon$$

when we want to make explicit the dependence of the density functions that describe the induced population partitions on the policies proposed by the parties.

Note that, for the mean voter rule, if the proposal of the parties satisfy  $x = (x_1, x_2) \in \Delta$ , then  $P_1(g_1^x) = P_2(g_2^x) = \mu$ , the observed mean of the overall population on  $A$ . Therefore,  $(\mu, \mu)$  is a fixed point of  $\phi(x) = (\phi_1(x), \phi_2(x)) = P(g_1^x(\alpha), g_2^x(\alpha))$ . We want to determine conditions under which the fixed point  $x(\mu)$  is unstable off diagonal. As the stability along the diagonal is immediate from the definition of the rule (from anywhere on  $\Delta$  the function  $\phi$  immediately maps to  $(\mu, \mu)$ ) we should simply need that the eigenvalues of the matrix

$$B(x) = \begin{pmatrix} \partial_{x_1} \phi_1(x) - 1 & \partial_{x_1} \phi_2(x) \\ \partial_{x_2} \phi_1(x) & \partial_{x_2} \phi_2(x) - 1 \end{pmatrix}$$

have different signs around  $(\mu, \mu)$ . In other words, the necessary condition for assumption 3 to hold is that

$$(1) \quad \lim_{x \rightarrow (\mu, \mu)} \det(B(x)) < 0$$

**Lemma 8.** The map  $\phi : X^2 \rightarrow X^2$  is continuous and differentiable.

**(Proof):** For  $\alpha$  and  $j = 1, 2$  fixed, the function  $g_j^x(\alpha)$  depends continuously on  $x$ . Thus,  $P_j(g_j^x)$  depends continuously on  $x$ . In addition, by Lemma 1 the mapping  $g_j^x(\alpha)$  is Fréchet differentiable in  $x$ . Since,  $P_j(g_j)$  is linear in  $g_j$  and continuous, the composition is also differentiable.

**Lemma 9.**

$$\lim_{x \rightarrow (\mu, \mu)} \partial_{x_i} \phi_j(x) = \frac{1}{\lambda_j} \int_A (\alpha - \mu) \left( \partial_{x_i} g_j^{(\mu, \mu)}(\alpha) \right) d\alpha$$

**Proof Lemma 9:** Note that

$$\begin{aligned} \partial_{x_i} \phi_j(x) &= \frac{1}{\left( \int_A g_j^x(\alpha) d\alpha \right)^2} \left( \int_A \alpha \partial_{x_i} g_j^x(\alpha) d\alpha \int_A g_j^x(\alpha) d\alpha - \int_A \alpha g_j^x(\alpha) d\alpha \int_A \partial_{x_i} g_j^x(\alpha) d\alpha \right) \\ &= \frac{1}{\left( \int_A g_j^x(\alpha) d\alpha \right)} \left( \int_A \alpha \partial_{x_i} g_j^x(\alpha) d\alpha - \phi_j(x) \int_A \partial_{x_i} g_j^x(\alpha) d\alpha \right) \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{x \rightarrow (\mu, \mu)} \partial_{x_i} \phi_j(x) &= \lim_{x \rightarrow (\mu, \mu)} \frac{1}{\left( \int_A g_j^x(\alpha) d\alpha \right)} \left( \int_A \alpha \partial_{x_i} g_j^x(\alpha) d\alpha - \phi_j(x) \int_A \partial_{x_i} g_j^x(\alpha) d\alpha \right) = \\ &= \frac{1}{\lambda_j} \int_A (\alpha - \mu) \left( \partial_{x_i} g_j^{(\mu, \mu)}(\alpha) \right) d\alpha \end{aligned}$$

and the Lemma follows.

We now retake the proof of the Proposition 7. When parties choose the proposals  $x = (x_1, x_2)$ , the agents affiliate to one or the other party, depending on which side of the line

$$z(t) = \frac{x_1 - x_2}{y_2 - y_1} t + \frac{y_1 + y_2}{2} - \frac{x_1^2 - x_2^2}{2(y_2 - y_1)}$$

they fall. For example,

$$g_1^x(\alpha) = \int_{-\infty}^{z(\alpha)} f(\alpha, \varepsilon) d\varepsilon$$

holds for  $x_1$  close enough to  $\mu$ . Hence,

$$(2) \quad \begin{aligned} \partial_{x_1} g_1^x(\alpha) &= f(\alpha, z(\alpha)) \partial_{x_1} z(t)|_{t=\alpha} = f(\alpha, z(\alpha)) \frac{\alpha - x_1}{y_2 - y_1} \\ \partial_{x_1} g_1^x(\alpha) &= f(\alpha, z(\alpha)) \partial_{x_1} z(t)|_{t=\alpha} = f(\alpha, z(\alpha)) \frac{\alpha - x_1}{y_2 - y_1} \end{aligned}$$

Furthermore,

$$(3) \quad \partial_{x_2} g_1^x(\alpha) = f(\alpha, z(\alpha)) \partial_{x_2} z(t)|_{t=\alpha} = f(\alpha, z(\alpha)) \frac{x_2 - \alpha}{y_2 - y_1}$$

which implies that

$$\partial_{x_1} g_1^{(\mu, \mu)}(\alpha) = f\left(\alpha, \frac{y_1 + y_2}{2}\right) \frac{\alpha - \mu}{y_2 - y_1} = -\partial_{x_2} g_1^{(\mu, \mu)}(\alpha)$$

Finally, since for the party 2 the relevant population density is

$$g_2^x(\alpha) = \int_{z(\alpha)}^{\infty} f(\alpha, \varepsilon) d\varepsilon$$

we get that

$$(4) \quad \begin{aligned} \partial_{x_2} g_2^{(\mu, \mu)}(\alpha) &= -\partial_{x_1} g_2^{(\mu, \mu)}(\alpha) = \partial_{x_1} g_1^x(\alpha) = \\ &= f\left(\alpha, \frac{y_1 + y_2}{2}\right) \frac{\alpha - \mu}{y_2 - y_1} \end{aligned}$$

Thus, applying the Lemma 9 we have to establish conditions under which

$$\begin{aligned} &\lim_{x \rightarrow (\mu, \mu)} \det \begin{pmatrix} \partial_{x_1} \phi_1(x) - 1 & \partial_{x_1} \phi_2(x) \\ \partial_{x_2} \phi_1(x) & \partial_{x_2} \phi_2(x) - 1 \end{pmatrix} = \\ \det \begin{pmatrix} \frac{1}{\lambda_1} \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha - 1 & -\frac{1}{\lambda_1} \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha \\ -\frac{1}{1-\lambda_1} \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha & \frac{1}{1-\lambda_1} \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha - 1 \end{pmatrix} = \\ &\left( \frac{1}{\lambda_1} \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha - 1 \right) \left( \frac{1}{1-\lambda_1} \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha - 1 \right) - \\ &\quad - \frac{1}{\lambda_1(1-\lambda_1)} \left( \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha \right)^2 = \\ &\quad 1 - \frac{1}{\lambda_1(1-\lambda_1)} \int_A (\alpha - \mu) \left( \partial_{x_1} g_1^{(\mu, \mu)}(\alpha) \right) d\alpha = \\ &\quad 1 - \frac{1}{y_2 - y_1} \frac{1}{\lambda_1(1-\lambda_1)} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha < 0 \end{aligned}$$

which clearly holds if  $(y_2 - y_1) < \frac{1}{\lambda_1(1-\lambda_1)} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha$ .

The boundary established by proposition 7 depends only on two bits of population statistics: the variance of ideal points in the observable ideological space of those citizens who would be indifferent between parties in the absence of ideological differences between them, and the relative size  $\lambda_1$  of the part of the population that exogenously prefers one party to another when there is no ideological difference between them:

$$\eta = \frac{1}{\lambda_1(1-\lambda_1)} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha$$

While, strictly speaking, exceeding the boundary does not guarantee the uniqueness of the convergent equilibrium, examples of the latter are not hard to find.

**Example 10.** Let the population distribution be uniform ( $f(\alpha, \varepsilon) = 1$  on  $[0, 1]^2$ ) and  $y_1 = \frac{1}{4} < \frac{3}{4} = \frac{1}{2}$ . The unique equilibrium in this case has  $x_1 = x_2 = \frac{1}{2}$  (show). Conversely, for any  $(y_1, y_2)$  such that  $|y_2 - y_1| < \frac{1}{12\lambda_1(1-\lambda_1)}$  divergent equilibria exist by our proposition.

Note that the number  $\eta$  in Proposition 7 depends only on some statistics of the population.

Since small exogenous differences between the parties implies that the citizens' membership decision is mostly determined by the observed policy differences (in particular, if  $y_2 = y_1$  we are reduced to the deterministic model), this result shows that the deterministic case is not isolated. Rather, in this case there is continuity: a

small amount of uncertainty about individual membership decision does not affect the existence of a divergent equilibrium.

It should be noted that the continuity result of Proposition 7 can be extended to policy rules other than the mean voter rule, though the precise boundary would be different. In particular, suppose that, instead of choosing the ideal point of the mean of its voter distribution, parties propose policies according to a different rule

$$P_j(g_j^x) = \frac{\int_A Q(\alpha)g_j(\alpha) d\alpha}{\int_A g_j(\alpha) d\alpha}$$

where  $Q : A \rightarrow A$  is a non-constant continuous mapping and denote the policy society as a whole would adopt as  $\chi = \int_A Q(\alpha)f_1(\alpha) d\alpha \in \text{int}(A)$ . Following the steps of the proof of proposition 7 we shall easily obtain the following boundary on the exogenous difference between parties that guarantees existence of divergent equilibria:

$$0 < |y_2 - y_1| < \frac{1}{\lambda_1(1 - \lambda_1)} \int_A (Q(\alpha) - \chi)(\alpha - \chi) f(\alpha, \frac{y_1 + y_2}{2}) d\alpha$$

which implies that, as long as  $\int_A (Q(\alpha) - \chi)(\alpha - \chi) f(\alpha, \frac{y_1 + y_2}{2}) d\alpha > 0$  for these rules, likewise, sufficiently small uncertainty about individual membership choices leads to the existence of divergent equilibria.

It is possible to use the same techniques to establish similar bounds for other rules, that do not belong to the class described above. For example, suppose parties instead use the median voter rule, i.e.  $P_j(g_j)$  is defined by

$$\int_{\{\alpha: \alpha \leq P_j(g_j)\}} g_j(\alpha) d\alpha = \frac{1}{2}$$

Clearly, what we are after is finding the fixed points of the mapping  $\phi : X^2 \rightarrow X^2$  given by an implicit equation

$$\int_{\{\alpha: \alpha \leq \phi_j(g_j^x)\}} g_j^x(\alpha) d\alpha = \frac{1}{2}$$

Denoting the median of the whole population distribution as  $m$ , it is not hard to observe that the point  $(m, m) \in \Delta$  is indeed the fixed point of this mapping, and, therefore, corresponds to a convergent equilibrium. As before, we would like to establish the (in)stability of  $\phi$  around the convergent fixed point, that is we would like to establish conditions under which inequality 1 holds.

Differentiating  $\phi$  implicitly with respect to  $x_i$  we obtain

$$g_j^x(\phi_j(x)) \partial_{x_i} \phi_j(x) + \int_{\{\alpha: \alpha \leq \phi_j(g_j^x)\}} \partial_{x_i} g_j^x(\alpha) d\alpha = 0$$

from which it follows that

$$\partial_{x_i} \phi_j(x) = - \frac{\int_{\{\alpha: \alpha \leq \phi_j(g_j^x)\}} \partial_{x_i} g_j^x(\alpha) d\alpha}{g_j^x(\phi_j(x)) \partial_{x_i} \phi_j(x)}$$

Taking the limit of the expression as  $x \rightarrow (m, m)$  and substituting from equations 2, 3 and 4 the formulas for  $\partial_{x_i} g_j^x(\alpha)$ , we obtain

$$\lim_{x \rightarrow (m, m)} \partial_{x_j} \phi_j(x) = \frac{\int_{\{\alpha: \alpha \leq m\}} (m - \alpha) f(\alpha, \frac{y_1 + y_2}{2}) d\alpha}{\lambda_j (y_2 - y_1) f_1(m)} = - \lim_{x \rightarrow (m, m)} \partial_{x_i} \phi_j(x), i \neq j$$

Thus, computing the determinant and rearranging terms we obtain the following proposition:

**Proposition 11.** Let  $n = 1$  and the parties be using the median voter rule. For any overall population distribution  $\nu$  on  $A \times E$  such that  $\alpha$  and  $\varepsilon$  are independent and a continuous population density function  $f(\alpha, \varepsilon)$  exists, if

$$\frac{1}{\lambda_1(1-\lambda_1)} \frac{1}{f_1(m)} \int_{\{\alpha:\alpha \leq m\}} (m-\alpha) f(\alpha, \frac{y_1+y_2}{2}) d\alpha > |y_2 - y_1| > 0$$

then, there exists a divergent equilibrium.

It should be noted that when the density of citizens's ideological viewpoints at the median point of the whole distribution is  $f_1(m) = 0$ , then the bound on  $|y_2 - y_1|$  explodes, as the minor changes of policies cause the intraparty medians to move at an infinite rate.<sup>3</sup> Another interesting observation is, that, at least for the uniform distribution of citizens over  $A \times E = [0, 1]^2$ , the boundary for the  $|y_2 - y_1|$  implied by the median voter rule (which, in this case is easily computed to be equal to  $\frac{1}{2}$  for *ex ante* identical parties with  $\lambda_1 = \lambda_2 = \frac{1}{2}$ ) is weaker than that for the mean voter rule (for which it is  $\frac{1}{3}$ ): as the policy difference between parties induces ideologically skewed memberships within each party, the medians move towards the edges faster than the means.

Finally, it should be noted that the divergent equilibria exist even when parties use distinct rules. Of course, the result is trivially true if there is no convergent equilibrium. Thus, if party 1 uses the mean voter rule, while party 2 uses the median voter rule a convergent equilibrium only exists if the mean equals the median for the overall population distribution  $f_1(\alpha)$ , i.e. if  $m = \mu$ . If  $m \neq \mu$  whatever equilibria exist (and some equilibria involving sorting over  $A \times E$  do exist from the results, for instance, of Caplin and Nalebuff 1997) would be divergent here. Still, even for the case when  $m = \mu$  we can guarantee existence of a divergent equilibria for  $|y_2 - y_1|$  small enough. In fact, using the inequality 1 we can establish that such equilibria exist in this case whenever

$$0 < |y_2 - y_1| < \frac{1}{\lambda_1} \int_A (\alpha - \mu)^2 f(\alpha, \frac{y_1 + y_2}{2}) d\alpha + \frac{1}{1 - \lambda_1} \frac{1}{f_1(\mu)} \int_{\{\alpha:\alpha \leq \mu\}} (\mu - \alpha) f(\alpha, \frac{y_1 + y_2}{2}) d\alpha$$

#### 4. CONCLUSIONS AND FURTHER RESEARCH

In this paper we introduce a model in which the citizens' party choice is determined both by the ideological difference between the parties and the unobserved non-ideological attitudes. As the membership choice is only incompletely determined by observed policy proposals, it may be interpreted as stochastic in this model. The party membership, in turn, determines party policy stances by means of intraparty preference aggregation rules.

In this context with two parties we show that, at least when the parties aggregate preferences by choosing ideal points of their mean (or median) voters, than, if parties are perceived by citizens as "similar" in the sense that the non-policy difference is small compared to the variance of the agents' preferences in the ideological

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<sup>3</sup>though, strictly speaking, the function  $\phi$  is not differentiable (not even necessarily continuous) in this case, the instability of the convergent equilibrium and the consequent existence of a divergent equilibrium can be easily shown using standard approximation techniques.

space, we are guaranteed existence of divergent equilibria even in an *ex ante* symmetric model. In this sense, the present stochastic model shows continuity with the deterministic endogenous platform model we studied earlier (Gomberg *et al.* 2004).

It remains to consider how the results extend to increasing the number of parties and of policy dimensions, as well as considering different party decision-making rules (including, possible strategic interaction in a democratic context).

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