Vote revelation: empirical characterization of scoring rules

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Abstract

In this paper I consider choice correspondences defined on an extended domain: the decisions are assumed to be taken not by individuals, but by committees and, in addition to the budget sets, committee composition is observable and variable. For the case of varying committees choosing over a fixed set of alternatives I provide a full characterization of committee choice structures that may be rationalized with sincere scoring (which has a natural interpretation if the number of alternatives is equal to 2). For the general case of variable budget sets a necessary implication of choice by sincere scoring is provided.

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1 Introduction

In November 2011 President Obama nominated Richard Taranto for a federal appeals court judgeship. This nomination was reported out of the Senate Judiciary Committee the following March. It was never voted on during the rest of the 112th Congress and was returned to the president. When the new Congress (it membership changed in the intervening election) assembled, Taranto was renominated in January 2013, reported out of the Judiciary Committee for the second time in February and, finally, confirmed by the full US Senate in March 2013. This fairly run-of-the-mill and uncontroversial nomination is only the latest (at the time of the writing) of the numerous similar actions concluded by the US Senate. Notably, it involved repeated decisions of varying groups of Senators on whether the nominee was deserving of the federal judgeship. In all cases we know the names of the Senators who could participate in taking the decisions and we know their decision as a group. In only one case, though, we have a record of individual roll call votes (the final 91 to 0 vote to confirm by the full Senate).[1] Consider now the job of a researcher trying to understand the Senate goings on from a database of observed committee and full Senate decisions. What sort of deductions would s/he be able to make possible to make about the preferences and behavior of the Senators?

Group, rather than individual, decisions may be all one can go by when studying settings other than legislatures (think, for instance, of monetary policy committees in central banks, courts - or, for that matter the old Soviet Politburo). We may know more or less about what happens inside the doors and inside the minds of the group members. Crucially, even when we know the formal decision rules the committee must act by (which may or may not be the case in practice), the "real" votes of committee members may be unknown, either because these are kept secret for policy reasons[2] or because they are not actually recorded[3], or because in many cases individual members

[1]All information taken from the web site of the Senate Judiciary Committee. The committee decisions were taken on a voice vote and no vote was taken by the full Senate in the 112th Congress.

[2]Whether voting records of central bank decision-makers should be public has been a subject of substantial controversy and research in recent years and the international practice has varied (see, for instance, Siber 2003, or Gersbach and Hahn 2008).

[3]Thus, in a legislature no formal vote may be taken on an issue since the parliamentary leaders know that it would fail anyway.

might are free to manipulate the formal record without impacting the actual decision. What we do usually know, though, is who was participating and what decision they took. As outside observers, we may want to have some questions answered from the observed data. Do the group members vote strategically or sincerely? Do they take into account preferences of and/or information possessed by their fellow committee members? If only committee decisions are made public, with votes and deliberations remaining secret, could we still test theories about the functioning of the committee?

It is well-known that not much could be said from a single observation of a committee choice or even of a series of choices made by a group of people, whose membership is unobserved. Formally, it has been known since McGarvey (1953) that, unless something is known about committee membership, *any* choice structure is consistent with even the most unsophisticated theory, such as, for instance, that the group members decide by simple majority, while voting sincerely and independently from each other, based on some unobserved individual preferences. However, as I try to establish in this paper, if observations of decisions taken by committees with *variable, but overlapping, membership* are available (as was the case with the nomination of Judge Taranto), one can use such data to reject, at least, this "naive" theory of committee decision-making.

The approach I use here, which was originally presented in my earlier note (Gomberg 2011)[4], is, in fact, quite standard, being based on the ideas of revealed preference and rationalizability, that have long been at the foundation of economic analysis. Ever since Houthakker (1950) it has been known that a simple consistency condition on choices (the Strong Axiom of Revealed Preference, SARP) is necessary and sufficient for being able to explain individual choices with rational preference maximization. Of course, this approach has long been a basis for the formal decision theory used by political scientists, as well as economists. That observations of group decisions themselves may be used to uncover both individual preferences and group decision rules is, however, frequently ignored. Even less studied is the potential for exploiting group membership data for such purposes.

The key point made in this paper is that *group decision data may be naturally richer* than would be usual in revealed preference models of individual choice: in addition to the record of choices from a given set of

[4]That note contains the basic statement of the problem, as well as some early examples and results.

alternatives, one has the committee membership at each decision point to consider. Hence, if one wants to test a given theory of how the committee works, one has more information to base it on. In the language of modern individual decision theory, group membership becomes a frame (as, for instance in Salant and Rubinstein 2008) Even if such data is incomplete (i.e., when not all possible observations might be in the data), one may hope that preference aggregation rules might have additional testable implications.

In this study I concentrate on just one particular class of theories about internal committee workings. I will assume that each committee consists of rational members who make a joint decision by using some scoring rule: that is, one of the voting rules, such as the simple "first past the post" plurality or the Borda Count, in which individuals are asked to provide each alternative with a numeric score (reflecting their preferences), the individual scores are added up and the alternative with the highest aggregate score is chosen. These rules have long been characterized by social choice theorists (see, for instance, Young 1975 or Myerson 1995) and are frequently used in practice. My objective is to formulate natural restrictions on observations implied by these rules when agents vote sincerely, without regard to other committee members

Of course, those axiomatizations take individual preferences as known. In contrast, in this paper preferences and votes are "revealed" from observations of group choices. One hopes to characterize the conditions under which these revealed scores are consistent with the proposed voting rule. For the simple case of varying committees choosing over a fixed pair of alternatives, in fact, such characterization turns out to coincide with the conditions for the existence of additive probability measures over a finite state space, representing a given binary relation "at least as likely as", established by Kraft *et al.* (1959). The necessary and sufficient condition for such a representation has a clear "SARP-like" acyclicity interpretation. This condition formally generalizes for any choice structure over a fixed finite set of alternatives (though the interpretation of the resultant condition may be harder). For the more general case of variable budget sets the natural necessary conditions of the "SARP"-type emerge,[5] though the complete characterization is, so far, unknown.

[5] Such as the "SARP-like" necessary condition, which was first formulated in my earlier note (Gomberg 2011).

1.1 Related Literature

The objective of this paper clearly comes close to the study of empirical content of sincere (vs. strategic) voting by Degan and Merlo (2009) and to Kalandrakis (2010) work on rationalizing individual voting decisions. In fact, as suggested above, if the formal decision rule is known, this work may be reinterpreted precisely as the test of voter sincerity: if I know how the votes are counted, violations of the conditions established here could only be interpreted as indications that the scores do not directly reflect rational individual preference. Thus, to the extent one maintains the assumption that voters are rational, sincere voting would be falsified in this case. Where my approach crucially differs from both Degan and Merlo (2009) and Kalandrakis (2010), however, in that I do not assume observability of individual votes (nor do I impose anything, in addition to rationality, on individual preferences), but try to infer votes from observing the group choice data.

Of course, it would be incorrect to claim, that using group choice data to obtain revealed-preference-like conclusions has never been suggested. Thus, for instance, when Blair *et al.* (1976) characterized such restrictions on choice structures as would derive from maximizing preferences that are merely acyclic, rather than transitive, this could, of course, be interpreted as characterizing choices made by committees of rational members with some of those members exercising veto power. The above-mentioned "non-falsification" result by McGarvey (1953) has been extended by Deb (1976) to certain voting rules other than majority, strengthening the point that without more data theories of internal committee rules may not be testable. However, when the size of the committee is known, restrictions implied by various group choice rules on the minimal cycle length in choices have been studied by Nakamura (1979).[6] In a different context, Peters and Wakker (1991) discussed empirical consequences of bargaining solutions, as did, for instance, Chambers and Echenique in a recent paper (2013)[7]. Characterization of the empirical consequences of such behavior for the household demands by Browning and Chiappori (1998) has been particularly influential.

However, the context in which the revealed preference approach has, perhaps, been most productive recently, is that of individual decision-making,

[6] See Chapter 3 of Austen-Smith and Banks (1999) for a textbook treatment.

[7] Notably, in that latter paper the authors demonstrate that, unless the threat point data is available and variable, the variation on the size of the surplus is not sufficient to distinguish among some major bargaining solution concepts.

in particular in modeling "boundedly rational" decision-making procedures different from the usual rational preference maximization. Here one might mention, among many others, Manzini and Mariotti (2007) work on "sequential rationalizability" or Masatlioglu and Ok (2005) study of choice with status-quo bias, both of which establish restrictions imposed on choices by distinct decision-making procedures. Other recent studies, such as Caplin and Dean (2011) and Caplin, Dean and Martin (2011) attempt to explore the restrictions that various "boundedly rational" procedures would impose on records that are somewhat more detailed than the usual choice data, though still plausibly observable: the insight that is also of key importance to the present research.

Perhaps the most obvious parallel to the present study in the new behavioral literature is presented by the "multi-self" models of decision-making, such as Kalai et al. (2002), Green and Hojman (2007) and Ambrus and Rozen (2011).[8] Thus, Green and Hojman's (2007) approach to individual choice behavior as resulting from aggregation of multiple rational rankings arising from distinct *motivations*, has most clear similarities with mine, in as much as they consider the use of actual scoring rules to do the aggregation. In fact this approach is formally analogous to the preference aggregation in a committee (with each of the "motivation" corresponding to a committee member). Somewhat similarly, Ambrus and Rozen (2011) consider behavior that can be explained by aggregation of multiple utility functions (those of different "selves" in one's mind). As both Green and Hojman (2007) and Ambrus and Rozen (2009) observe, without any restrictions on the number of motivations/utility functions inside one's mind, most aggregation rules would not provide falsifiable restrictions on the resulting choices. However, if the cardinality of the set of "selves" were to be known, such restrictions would, indeed, emerge.[9]

One key distinction between Green and Hojman's (2007), Ambrus and Rozen (2011) and the present paper is, in fact, in the sort of the data that is available for testing the theory. Whereas the "selves" inside one's mind are not directly observable and may only be inferred from individual behavior, in the context of collective decisions the group membership might itself

[8] Another related result is provided by Bogomolnaia and Laslier (2007), who establish results on a number of policy space dimensions that would suffice for representing a given preference order with a Euclidean distance.

[9] This would be the interpretation most similar to the Nakamura (1979) results mentioned above.

reasonably constitute part of the observed data. Not only that, but it may be natural to assume that one could observe variation not only of the set of available alternatives, as is standard in the revealed preference literature, but also of group composition itself: for instance, we may observe decisions on a given issue of different parliamentary committees and subcommittees, the membership of which is known, or use changes in court or monetary committee composition.

Another related problem has been addressed in the psychological literature by Falmagne (1978; introduced to economists by Barbera and Pattanaik in 1986), who, effectively, considered restrictions on the probability distribution of choices generated from a fixed choice set by an unobserved decision-maker randomly drawn from a general population with an unknown distribution of heterogenous preference types. In difference with the unobserved random dictator of Falmagne (1978), in this paper I assume that we do observe not only the (possibly non-unitary) set of decision-makers, but also variations in it.

The rest of this paper is organized as follows. In section two I provide the basic model set-up. In section three I consider the simple case of two alternatives and provide a characterization of the restrictions on the committee choice structures that make them consistent with choice by scoring In section four I extend the analysis to the case of three or more alternatives. Section five concludes.

2 Basic Set-up

In this section I closely follow my earlier note (Gomberg 2011). Consider a finite set $N = \{1, 2, ...n\}$ of agents and a finite set $X = \{x_1, x_2...x_m\}$ of alternatives. A set of alternatives to be considered by a *committee* $S \in 2^N \backslash \{\varnothing\}$ is $B \in 2^X \backslash \{\varnothing\}$; following the standard terminology of individual choice theory, I shall call B the *budget set*. If a committee S is offered a choice from the budget set B the committee choice is recorded as $\varnothing \neq C(B, S) \subset B$. The *committee choice structure* is defined as a pair $(\mathcal{E}, C(.,.))$ where $\mathcal{E} \subset 2^X \backslash \{\varnothing\} \times 2^N \backslash \{\varnothing\}$ is the record of which budget sets where considered by which committees and $C : \mathcal{E} \to X$, such that $C(B, S) \subset B$ is the non-empty-valued choice correspondence, recording committee choices.

In order to explain observed committee choice structures, I shall, in general, assume that each agent $i \in N$ has rational (complete and transitive) preferences $\succsim_i$defined over X. The committee choice structure provides a record of observed committee choices, which may be used by an observer to deduce the preference profiles and the preference aggregation rules the committee uses. In this paper I concentrate on a particular class of such rules: the scoring rules, a class that includes such distinct procedures as the plurality rule (in which the winner is an alternative that is chosen by the largest number of voters), the Borda count (in which alternatives get assigned the most points for being someone's top choice, a point less for being a second choice, etc., the scores get summed up over all the voters and the alternative with the largest score wins), or the Approval vote (in which an individual is allowed to mark alternatives as acceptable or unacceptable, and the alternative which has been marked as acceptable by the largest number of voters gets chosen).

In general, I shall assume that agents are non-strategic, in that they ignore who else is in the committee (as noted above, the conditions I am deriving here might, if the formal rule is observable, be viewed as empirical implications of sincere voting itself). However, I shall allow the votes to depend on the budget sets under consideration (as would be the case in a sincere Borda count). Thus, if the set of alternatives B, a vote of agent $i \in S$ is a function $v_i^B : B \to \mathbb{R}$. Vote independence from committee membership is, in fact, an extremely strong condition that not only would be inconsistent with strategic voting, but would also eliminate possible vote variations due to interdependent preferences or differential information.

Given a vote from each of its members a committee S chooses an alternative that gets the highest score

$$C^{scoring}(B,S) = \arg\max_{x \in B} \sum_{i \in S} v_i^B(x)$$

where $\sum_{i \in S} v_i^B(x)$ is called the *score* received by an alternative $x \in B$ in voting by committee S. Such a choice structure is said to be generated by the scoring rule.

Following Myerson (1995) I shall allow agents to submit votes that are distinct from reporting their preference orderings. In fact, for the purposes of defining a scoring rule one does not need to assume that the votes themselves derive from rational preferences. All the scoring rules require agents to do, is to report a ranking of alternatives in B by means of their votes $v_i \in \mathbb{R}^k$.

In general such a ranking may not necessarily represent a rational preference (and thus, for instance, could be inconsistent over the different budget sets B). Nevertheless I shall concentrate on voting that, indeed, can be viewed as a sincere representation of individual preferences. Formally, given a rational preference profile $\succsim = (\succsim_1, \succsim_2, ..., \succsim_n)$ I shall say that a committee vote v_i^B is *strictly consistent with preferences* if $x \succsim_i y$ if and only if $v_i^B(x) \geq v_i^B(y)$. I shall say that a committee vote v_i^B is *weakly consistent with preferences* if $x \succsim_i y$ implies $v_i^B(x) \geq v_i^B(y)$. The obvious reason why both consistency notions are of interest here is that though in some scoring rules (such as the Borda count) allow agents to submit what amounts to utility functions (i.e., actual representations of their preferences), other rules do not. Thus, under the plurality rule (in which an agent can only affirmatively vote for one alternative and can make not distinctions about the rest) even a sincere vote would only be weakly consistent with individual preferences as long as there are, at least, 3 alternatives.

If a committee choice structure is such that for any $(B, S) \in \mathcal{E}$

$$C(B, S) = C^{scoring}(B, S)$$

where the votes are consistent with preferences for some rational preference profile $\succsim$. I shall say that $\succsim$ rationalizes $(\mathcal{E}, C(.,.))$ via a scoring rule.

It should be noted, that unless the choice structure is extended by allowing observing variations in committee membership, scoring rules would, at first glance, appear particularly unpromising from the standpoint of this research: it would seem that nearly every possible committee decision could be explained by some sort of scoring applied to an unobserved preference profile of a fixed committee. Thus, if one defines, in the spirit of Salant and Rubinstein (2008) work on the choice with frames, the choice correspondence as

$$C(B) = \{x : x \in C(B, S) \text{ for some committee } S\}$$

not much structure appears be imposed on $C_c(.)$, though it would follow, for instance, from Ambrus and Rozen (2011) that some restrictions may be derived from the cardinality of N, if that is observed. However, it turns out that more can be said if committee membership and its variations are observed.

3 Revealed Scoring: The case of two (fixed) alternatives

I shall first consider the simple case, in which the number of alternatives is equal to 2. In this case, scoring may be interpreted as weighted majority vote in which, for the purposes of this model, both the individual preferences and weights are unobservable. In this case the only interesting budget set is $B = X = \{x_1, x_2\}$, so that the entire observable variation comes from the committee membership. The choice C here is a mapping from a subset of the set of observed committees $\mathcal{E} \subset 2^N \setminus \{\varnothing\}$, that may take one of only three values: $\{x_1\}$, $\{x_2\}$ or $\{x_1, x_2\}$.

It is clear that not every such committee choice structure would be rationalizable with sincere scoring. Crucially, the notion of sincere scoring studied here implies that each individual's votes are independent of the committee composition. Hence, if we ever observe that for two disjoint committees $S \cap T = \varnothing$ we have $C(S) = C(T) = x_i$ it must, indeed, follow that $C(S \cup T) = x_i$. This, property, introduced, for instance, in characterizations of scoring rules by Smith (1973) and Young (1975) is usually known as the *reinforcement* axiom. Clearly, reinforcement must be a necessary condition for the rationalizability here desired. But the scoring has an even stronger implication for the actual scores that committees assign to alternatives: the score difference between the alternatives must be added up if two disjoint committees are joined.

In fact, if sincere scoring is the rule used, the difference w between the scores assigned to x_1 and to x_2 by the committee S

$$w(S) = \sum_{i \in S} v_i^B(x_1) - \sum_{i \in S} v_i^B(x_2)$$

will define a (signed) measure on the finite measurable space $\left(N, 2^N\right)$, as long as one naturally sets $w(\varnothing) = 0$, since $w(S \cup T) = w(S) + w(T) - w(S \cup T)$ for any two committees $S, T \in 2^N$.

However, we do not observe the actual scores or their differences, but only choices, which correspond to the sign of w. Defining $\mathcal{E}^* = \mathcal{E} \cup \varnothing$ it may be convenient to summarize our observations with a function $f : \mathcal{E}^* \to \{-1, 0, 1\}$ defined by the

$$f(S) = sign(w(S)) = \begin{cases} -1, \text{ if } C(S) = \{x_2\} \\ 0, \text{if } C(S) = \{x_1, x_2\} \text{ or } S = \varnothing \\ 1, \text{if } C(S) = \{x_1\} \end{cases}$$

This function f is, of course, non-additive. If, however, we can, consistently with it, assign individual vote differences w_j to each individual in such a way that

$$sign(w(S)) = sign\left(\sum_{j \in S} w_j\right) = f(S)$$

, we shall obtain a scoring-based theory that would explain how the observed choice structure arose!

Fortunately, it turns out that this problem is closely related to well-established problems in utility theory. In fact, a very similar mathematical problem emerges if one considers the question of when could a binary relation "at least as likely as" over a finite states space be represented by a probability measure, which has been posed and solved by Kraft et al. (1959). The following example, which is, essentially, a reinterpretation for the present setting of the one they constructed, implies that the reinforcement alone, though necessary, is not sufficient for such a theory to be possible.

Example 1 *Suppose* $N = \{1, 2, 3, 4, 5\}$ *and* $f(\{4\}) = f(\{2, 3\}) = f(\{1, 5\}) = f(\{1, 3, 4\}) = 1$ *whereas* $f(\{1, 3\}) = f(\{1, 4\}) = f(\{3, 4\}) = f(\{2, 5\}) = -1$*. The example does not violte reinforcement. However, it is not hard to see that this set of choices is not consistent with sincere scoring, as it would imply that* $2w_1 + w_2 + 2w_3 + 2w_4 + w_5$ *is simultaneously positive and negative!*

Consequently, a stronger condition, which I shall call *strong reinforcement,* is required, which is analogous to strong additivity of Kraft et al. (1959). Following Fishburn (1986) it can be presented as follows. Consider two collections (of equal cardinality) of committees $\mathbf{S} = (S_1, S_2, ..., S_m)$ and $\mathbf{T} = (T_1, T_2, ..., T_m)$. Note, that an empty set is taken here as a possible committee and that a committee might be repeated several times within a collection. Denote as $n_j(\mathbf{S})$ the number of committees in the collection $\mathbf{S}$ that individual j is included in. We say that $\mathbf{S} \approxeq \mathbf{T}$ if for each individual $j \in N$ $n_j(\mathbf{S}) = n_j(\mathbf{T})$.

- The choice correspondence C satisfies *strong reinforcement* if for each pair of committee collections $\mathbf{S}, \mathbf{T}$ such that $\mathbf{S} \approxeq \mathbf{T}$ if $f(S_i) > f(T_i)$ or $f(S_i) = f(T_i) = 0$ for $i = 1, 2, ..., m-1$ then not $f(S_m) > f(T_m)$.

It should be noted that strong reinforcement is indeed a strong property, which implies a number of desirable conditions of the choice structures. Thus, it can be easily seen to imply the reinforcement property itself. It turns out that, in fact, this condition characterizes choice structures that can be explained with sincere scoring.

Theorem 1 *A committee choice structure* $(\mathcal{E}, C(.,.))$ *may be generated by a scoring rule strictly consistent with rational preferences if and only if the choice structure satisfies strong reinforcement.*

Proof. The necessity part is straightforward, since if it were not the case, there would exist a pair of committee collections $\mathbf{S} \approxeq \mathbf{T}$ such that $f(S_i) > f(T_i)$ or $f(S_i) = f(T_i) = 0$ for all $i = 1, 2, ..., m-1$ and $f(S_m) > f(T_m)$ However, as $f(S_i) = sign(w(S_i)) = sign\left(\sum_{j \in S_i} w(\{j\})\right)$ it follows that $\sum_{j \in S_i} w_j > \sum_{j \in T_i} w_j$ or $\sum_{j \in S_i} w_j = \sum_{j \in T_i} w_j = 0$ for $i = 1, 2, ...m-1$ and $\sum_{j \in S_m} w_j > \sum_{j \in T_m} w_j$, which, if we some across the committees in each collection, in turn would imply that $\sum_{j \in N} n_j(\mathbf{S}) w_j > \sum_{j \in N} n_j(\mathbf{T}) w_j$.

The proof of sufficiency closely follows that of Theorem 4.1 in Fishburn (1970). If all committees make the same choice, the theorem is trivially true, therefore, I shall henceforth assume that there exists at least one pair of committees $(S, T) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $f(S) > f(T)$ Let $K \in \mathbb{N}$ be equal to the number of distinct committee pairs $S, T \in \mathcal{E}$ $(S \neq T)$ such that $f(S) > f(T)$ and $M \in \mathbb{Z}_+$ be equal to one half of the number of committee pairs $(S, T) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $f(S) = f(T) = 0$. Note that the latter includes committee pairs of the form $(S, \varnothing)$ and $(\varnothing, T)$. Clearly, $K + M \leq 2^{2n} < \infty$.

For each committee S let the indicator function

$$1_S(j) = \begin{cases} 1 \text{ if } j \in S \\ 0 \text{ if } j \notin S \end{cases}$$

Clearly, if for each of the first $k = 1, 2..., K$ committee pairs S^k, T^k defined above we may write

$$\sum_{j=1}^{n} w_j a_j^k > 0$$

and for each of the following $k = K+1, K+2, ...K+M$ committee pairs S^k, T^k we may write

$$\sum_{j=1}^{n} w_j a_j^k = 0$$

where $a_j^k = (1_{S^k}(j) - 1_{T^k}(j)) \in \{-1, 0, 1\}$, and the weights $\sum_{j=1}^{n} w_j$ may be interpreted as a "reconstruction" of the individual vote difference consistent with the observed choice structure (note, in particular, that this would imply that $\sum_{j=1}^{n} w_j 1_S(j) = 0$ for every S such that $f(S) = 0$).

Suppose this is impossible. Then by Theorem 4.2 in Fishburn (1970), known as the Theorem of the Alternative, there must exist a collection of numbers $r_{k,}$ $k = 1, 2, ..., M+K$, such that the first K of these are non-negative and not all zero so that for every $j = 1, 2, ..., n$

$$\sum_{k=1}^{K+M} r_k a_j^k = 0$$

In fact, since all a_j^k are rational by construction, all r_k may be chosen to be integers. If for some $k > K$ there is an $r_k < 0$ one may replace a_j^k with $-a_j^k$ to make it positive (this is possible since if $f\left(S^k\right) = f\left(T^k\right)$ one may interchange S^k and T^k). Consider now two committee collections $\mathbf{S}$ and $\mathbf{T}$ such that each committee S^k is repeated r^k times in $\mathbf{S}$ and each committee T^k is repeated r^k times in $\mathbf{T}$. By construction the cardinality of each committee collection is equal to $\sum_{k=1}^{K+M} r_k$ and from the preceding equation it follows that the number of times each individual is included in committees in each collection is

$$n_j(\mathbf{S}) = \sum_{k=1}^{K+M} r_k 1_{S^k}(j) = \sum_{k=1}^{K+M} r_k 1_{T^k}(j) = n_j(\mathbf{T})$$

and, hence $\mathbf{S} \approxeq \mathbf{T}$. But by construction we have $f\left(S^k\right) \geq f\left(T^k\right)$ for all $k = 1, 2, ...K+M$, with the first K inequalities strict. Hence, the strong reinforcement of the committee choice structure is violated. QED

4 Three or more alternatives

4.1 Constant budget set

If there are three or more alternatives the problem cannot be reduced to that of an existence of a single measure on the committee space. Nevertheless, as long as all the committees are facing the same choice problem (i.e., the budget set B is not varied), the linear structure of the scoring rules utilized in the previous section allows for a very similar formulation.

Our basic objective remains the same: to find vote scores for each individual that would explain the observed committee choices. Notably, once there are at least three alternatives, we now will have to avoid "scoring cycles", as the following example shows.

Example 2 *(Gomberg 2011) Consider the budget set $B = \{a, b, c\}$ and the four disjoint committees S_1, S_2, S_3 and T. Let $C(B, S_1) = a$, $C(B, S_2) = b, C(B, S_3) = c$, $C(B, S_1 \cup T) = b, C(B, S_2 \cup T) = c, C(B, S_3 \cup T) = a$. It is not hard to see that this implies that $bP_{B,T}cP_{B,T}aP_{B,T}b$ which, of course, implies and impossible cycle: committee T should be giving alternative b a higher score than alternative c, alternative c a higher score than alternative a, and alternative a the higher score than alternative b, which is impossible.*

As the example above suggests, the scores may be "revealed" through observed committee choices (the revelation idea first introduced in Gomberg 2011). As, for the rest of this section, the budget set is fixed, I shall only consider variations in the committee membership $S \subset N$.

- *Direct revelation.* For each $S \in \mathcal{E}$ a pair of nested binary relations $P_S^* \subset R_S^*$ on B is defined by

  (i) let $x \in C(B, S)$ then xR_S^*y for any $y \in B$

  (ii) let $x \in C(B, S)$ and $y \notin C(B, S)$ for some $y \in B$ then xP_S^*y

This constitutes a record of direct preference revelation: if an alternative is chosen, it implies it received at least as high a score as any other feasible alternative and a strictly higher score than any feasible alternative not chosen.

Consider the total set of observations we have. If our theory is correct and this choice is rationalized with scoring, in the actual vote count each observation of xP_S^*y it must have been obtained from $\sum_{i\in S} v_i^B(x) > \sum_{i\in S} v_i^B(y)$ and each xR_S^*y from $\sum_{i\in S} v_i^B(x) \geq \sum_{i\in S} v_i^B(y)$. These are, of course, linear inequalities. In fact, the set of all "revealed scoring" statements must have been generated by a system of linear inequalities, which would have to hold simultaneously for the rationalization to be possible.

Let the cardinality $\#B = m$. Consider a vector $w = (w_1, w_2, ..w_n, w_{n+1}, ...w_{2n}, ...w_{nm}) \in R_+^{nm}$ where w_{kn+j} corresponds to the reconstructed vote that agent k emits for alternative j. As in the previous section, I shall consider each revealed scoring statement (taking care to track the committee by which it has been generated). As the total number of such statements is finite, let K be the number of strict statements xP_S^*y and M be one half of the rest.

Consider a list of all such revealed scoring pairs If the kth pair is $x_kP_S^*x_l$ (for the first K elements of the list) or $x_pR_S^*x_r$ (for the rest) then one can define $a_j^k = 1$ for all $j = p + ms$, where $s \in S$ $a_j^k = -1$ for all $j = r + ms$, where $s \in S$, and $a_{ij} = 0$ otherwise. As in the case of two alternatives, if for each of the first $k = 1, 2..., K$ revealed preference scoring relations defined above we may write

$$\sum_{j=1}^{n} w_j a_j^k > 0$$

and for each of the following $k = K+1, K+2, ...K+M$ revealed scoring relations we may write

$$\sum_{j=1}^{n} w_j a_j^k = 0$$

we would rationalize the observed choice structure.

As in the previous section, the Theorem of the Alternative allows one to restate the problem of existence of a solution to this system of inequalities as a problem of existence of a solution to the equation

$$\sum_{k=1}^{K+M} r_k a_j^k = 0 \qquad (*)$$

where $(r_1, r_2, ...r_{K+M}) \in \mathbb{Z}^{K+M}$ with the first K terms non-negative and not all equal to zero.

As in the case of two alternatives, this condition is, in fact, necessary and sufficient for the existence of rationalization by scoring, though it is harder to get its intuitive interpretation. A greater feeling for its implication may be obtained if we reformulate a necessary implication of it in a more familiar "revealed preference" form (as earlier proposed in Gomberg 2011).

Consider, for instance, the "indirect revealed scoring" implied by the reinforcement property of the scoring rules (which, as noted above states that if two disjoint committees make the same choice from a given budget set, so should their union). We can then define the following .

- *Reinforcement*[10]

  The binary relations $P_S \subset R_S$ on B are defined by

  (i) xP^*y implies xPy, xR^*y implies xRy,

  (ii) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \cap T = \varnothing$, xR_Sy and xR_Ty imply that $xR_{S\cup T}y$

  (iii) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \cap T = \varnothing$, $xP_{B,S}y$ and $xR_{B,T}y$ imply that $xP_{S\cup T}y$

  (iv) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \subset T(T\backslash S \neq \varnothing)$, xP_Sy and yR_Tx imply that $yP_{T\backslash S}x$

  (v) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \subset T(T\backslash S \neq \varnothing)$, xR_Sy and yP_Tx imply that $yP_{T\backslash S}x$

With this in mind we may now define a simple acyclicity condition, motivated by the example above:

Axiom 1 (Committee Axiom of Revealed Preference (CARP)) [11]
For any $S \in 2^N \backslash \{\varnothing\}$ *and any* $x_1, x_2, ...x_n \in B$,
$x_1R_Sx_2, x_2R_Sx_3...x_{n-1}R_Sx_n$ *implies* $\urcorner(x_nP_Sx_1)$

It is straightforward to see that CARP is, in fact, implied by scoring

Proposition 1 *A committee choice structure* $(\mathcal{E}, C(.,.))$ *may be generated by a scoring rule strictly consistent with rational preferences only if the implied* R_S *and* P_S *satisfy CARP for each* $S \in 2^N \backslash \{\varnothing\}$.

[10]Note that example 1 above shows that a stronger indirect extension could be imposed here. However, reinforcement is more intuitive, so I stick to it as a necessary implication of rationalizability.

[11]The naming suggestion for this axiom, originally introduced in Gomberg (2011), belongs to Norman Schofield

4.2 Variable budget sets: Necessary conditions

If the budget sets are variable, however, there is no obvious way of restating this as a solution to a linear system. The problem is that reconstructed scores may vary depending on the budget set in question (think of the number of points an alternative gets from a given individual under Borda Count in different budget sets), but such scores should still be consistent with an underlying rational preference. Maintaining the requirement of strict consistency of votes with an underlying rational preference relation, I shall try to "reveal" as much as possible about such individual votes.

Naturally, the condition (*) would still have to hold for each budget set, but it is no longer sufficient for rationalizabitliy with scoring. A further necessary implication may be obtained by using the direct score revelation and its extension by reinforcement defined above.

However, the budget set variation provides us with further information that allows one to make inferences about individual preferences from either direct or indirect observations of singleton coalitions. If the agents votes are strictly consistent with underlying preferences, then if an individual is every revealed to give a greater score to one alternative than to another, this should be maintained in all budget sets. I will therefore extend the revealed score relations and define the individual revealed preference relation P_i as follows:

- *Individual preference revelation for strictly consistent scoring*

  (i) If $xR_{B,\{i\}}y$ for some $B \in 2^X \backslash \{\varnothing\}$ then $xR_{D,\{i\}}y$ for any D s.t. $x, y \in D$ and xR_iy

  (ii) if $xP_{B,\{i\}}y$ for some $B \in 2^X \backslash \{\varnothing\}$ then $xP_{D,\{i\}}y$ for any D s.t. $x, y \in D$ and xP_iy

Once the binary relations $R_{B,S}$ and $P_{B,S}$ are thus extended (including further extensions by reinforcement, if possible), we may formulate a stronger version of CARP:

Axiom 2 (CARP*)

For any $B \in 2^X \backslash \{\varnothing\}$, any $S \in 2^N \backslash \{\varnothing\}$ and any $x_1, x_2, ...x_n \in B$, $x_1R_{B,S}x_2, x_2R_{B,S}x_3...x_{n-1}R_{B,S}x_n$ implies $\urcorner(x_nP_{B,S}x_1)$

It should be noted that, taking $B = X$, as long as individual preference revelation is taken into account, CARP implies the usual Strong Axiom of Revealed Preference (SARP) for the individual preference revelation.

It is clear that CARP would have to hold if a committee of rational individuals is deciding by sincere votes using a scoring rule, since otherwise we'd have to accept either cycles in individual preferences or in group scores (as in the example above). Hence, the next result follows immediately from the construction.

Proposition 2 *A committee choice structure* $(\mathcal{E}, C(.,.))$ *may be generated by a scoring rule strictly consistent with rational preferences only if the implied* $R_{B,S}$ *and* $P_{B,S}$ *satisfy CARP* for each* $B \in 2^X \setminus \{\varnothing\}$ *and each* $S \in 2^N \setminus \{\varnothing\}$.

The content of individual preference revelation, however, would be different if the scoring is only weakly consistent with preferences. In this case the individual preference revelation is more limited:

- *Individual preference revelation for weakly consistent scoring*

  if $xP_{B,\{i\}}y$ for some $B \in 2^X \setminus \{\varnothing\}$ then $xR_{D,\{i\}}y$ for any D s.t. $x, y \in D$ and xP_iy

Though this more limited extension of the revealed scoring relation may be used to define a CARP as in the case of strictly consistent scoring (I shall call it CARP**), the latter would no longer imply the SARP for individual revealed preference, which would have to be assumed directly

Axiom 3 (Strong Axiom of Revealed Preference (SARP)) *For any* $i \in N$ $x_1P_ix_2....P_ix_n$ *implies* $\neg x_nP_ix_1$

Proposition 3 *(Gomberg 2011) A committee choice structure* $(\mathcal{E}, C(.,.))$ *may be generated by a scoring rule weakly consistent with rational preferences only if the implied* $R_{B,S}$ *and* $P_{B,S}$ *satisfy CARP** for each* $B \in 2^X \setminus \{\varnothing\}$ *and each* $S \in 2^N \setminus \{\varnothing\}$ *and* P_i *satisfies SARP for each* $i \in N$.

5 Conclusions and further research

This paper introduces the notion of a committee choice structure and establishes a necessary and sufficient condition for such a choice structure to be rationalizable via scoring rules when the committees decide over the fixed budget set, with a natural interpretation for the case of two alternatives. For the case when budget sets vary, so. far it has been possible to establish a set of properties of committee choice structures that are necessary consequences of sincere scoring-based committee decisions It remains to see if this could be strengthened to a concise sufficient condition for rationalizability with scoring. An interesting further extension of the model would be to consider the consequences of particular scoring rules, such as plurality, approval or the Borda Count.

In terms of practical application, this paper provides conditions on the choice structures that would have to be violated for models more complicated than "sincere scoring" being possible to test. It should be noted that "sincerity", as defined here, is simply a statement that voters always maintain the same ranking of alternatives and do not change their scores based on the identity of other people in the committee they are a part of. Of course, if voters behave in this way, it is impossible to distinguish a possible case of "strategic" voting from simply following a fixed preference relation. However, in many environments following one's preferences would imply a violation of sincerity as here defined. Thus, for instance, if voters have interdependent preferences with other committee members, or receive noisy signals about a common value of different alternatives, then they may change their behavior based on the identities of other committee members. Hence, choices of committees, consisting of such voters, even if they use a scoring rule, would likely violate conditions derived here. This suggests, that the approach in this paper may be used to develop tests for presence of preference interdependence or common value in voting settings when only committee decisions and memberships are observed.

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