Revealed votes

Andrei Gomberg*
ITAM

January 21, 2018

Abstract

In this paper I consider choice correspondences defined on a novel domain: the decisions are assumed to be taken not by individuals, but by committees, whose membership is observable and variable. In particular, for the case of two alternatives I provide a full characterization of committee choice structures that may be rationalized with two common decision rules: unanimity with a default and weighted majority.

Keywords: choice, decisions, rationalizability, committees
JEL Codes: D02, D03, D71, D72

*An earlier verision of this paper was previously circulated under the title *Vote revelation: empirical consequences of scoring*. I would like to thank Attila Ambrus, David Austen-Smith, Juan Bertomeu, Rajat Deb, Diego Dominguez, Federico Echenique, Konrad Grabiszewski, Paola Manzini, Marco Mariotti, César Martinelli, Yusufcan Masatlioglu, Nicolas Melissas, Martin Osborne, Michael Richter, Ariel Rubinstein, Tridib Sharma, Levent Ulku, Radovan Vadovic and the participants of the ICOPEAI 2010 conference at Vigo (which gave rise to the volume in which my earlier note - Gomberg 2011 - on a related topic was published), 2011 APET conference in Bloomington, IN, 2011 SAET conference in Faro, Portugal, the 2012 World Congress of the Game Theory Society, the NES 20th Anniversary Decision Theory Workshop in Moscow and the Winter 2013 Computational Social Choice Workshop in Singapore, 2013 ITAM Decision Theory Workshop, 2013 NYU Alumni Conference and the 2014 Social Choice and Welfare Meetings, as well as the seminar participants at CEFIR, ITAM and the SMU for valuable ideas. My particular gratitude goes to Andrew Caplin, conversations with whom helped me formulate this problem. Financial support of Associación Mexicana de Cultura, A.C. is gratefully acknowledged. CIE–ITAM, Camino Santa Teresa 930, 10700 México DF, México. Phone: +52 55 56284197. Fax: +52 55 56284058, e-mail: gomberg@itam.mx

1 Introduction

Ever since Houthakker (1950) it has been known that a simple consistency condition on choices (the Strong Axiom of Revealed Preference, SARP) is necessary and sufficient for being able to explain individual choices with rational preference maximization. Of course, this approach has long been a basis for the formal decision theory used by political scientists, as well as economists. That observations of group decisions themselves may be used to uncover both individual preferences and group decision rules is, however, frequently ignored. Even less studied is the potential for exploiting group membership data for such purposes.

In this paper I apply the methodology of revealed preference to data consisting of choices made by a number of committees with overlapping membership. This allows me to define empirical consequences of two group-decision rules. In the first of these, one of the alternatives is marked as the default, which can only be overcome by a unanimous vote of all the group members. The other rule is the weighted majority (with individual group member weights unknown to the observer). I provide a complete characterization of each of these rules on my conjectured data set, by formulating the necessary and sufficient conditions for these rules to be consistent with the outcomes observed. I then expand the choice set facing the committees to more than two alternatives and extend the characterization of the weighted majority rule to this environment, while reinterpreting it as a more general *scoring* rule.

It has been long established that data on choices of a single committee from varying subsets of alternatives does not provide much in terms of testable implications of common group decision rules. Thus, McGarvey (1953) has demonstrated that, unless something is known about committee membership, *any* choice structure is consistent with the group members deciding by simple majority, while voting sincerely and independently from each other, based on some unobserved individual preferences (this result was later extended by Deb in 1976 to a large class of voting rules). Hence, the usual approach of varying the set of alternatives, while keeping committee membership constant is fruitless: given such a data set, many voting procedures within a committee simply have no testable implications.[1] However,

[1] These results apply when nothing is known about the group composition. When the size of the committee is known, there exist restrictions implied by various group choice rules on the minimal choice cycle length, which have been studied since Nakamura (1979).

as I try to show in this paper, if a data set of decisions taken by committees with *variable, but overlapping, membership* are available, the same voting procedures would impose a definite structure on such observations.

Group, rather than individual, decisions may be all one can go by in a variety of settings, such as legislatures, monetary policy committees in central banks, shareholder meetings, or courts. We may know more or less about what happens inside the doors and inside the minds of the various individuals forming a committee, the actions of which we observe. Crucially, even when we know the formal decision rules a committee must act by, the true votes of committee members may be unknown, either because these are kept secret for policy reasons[2] or because they are not actually recorded[3], or because in many cases individual members might be free to manipulate the formal record without impacting the actual decision. What we do usually know, though, is who was participating and what decision they collectively took. As outside observers, we may want to have some questions answered from the observed data. Were all members of a group treated equally, or did some of them possess greater weight? Were the various alternatives treated symmetrically, or were some of them privileged (as, for instance, a status quo could be in comparison with a reform proposal)? Did the group members vote strategically or sincerely? Did they take into account preferences of and/or information possessed by their fellow committee members? If only committee decisions are made public, with votes and deliberations remaining secret, could we still test theories about the functioning of the committee?

Committees with overlapping membership having to repeatedly decide over the same issue are not so rare in practice. Think, for instance, of the route that a bill or a nomination must pass to obtain approval by a legislative chamber, where it would have to be considered by one or many committees and subcommittees, all before coming up to the floor for the final vote. Similarly, in a number of Latin American countries, where a judicial decision does not always constitute legal precedent, courts have to repeatedly decide on constitutionality of the same law, as applied to different plaintiffs (such cases are known as *amparo* in Mexico[4] and *tutela* in Colombia). In the par-

[2]Whether voting records of central bank decision-makers should be public has been a subject of substantial controversy and research in recent years and the international practice has varied (see, for instance, Sibert 2003, or Gersbach and Hahn 2008).

[3]Thus, in a legislature no formal vote may be taken on an issue since the parliamentary leaders know that it would fail anyway.

[4]See, for instance, Vargas (1996) for a discussion of the role - and lack of precedential

ticular case of Colombia, the nine-member Constitutional Court has adopted a peculiar system of case designation, in which *tutela* decisions are *randomly* assigned to overlapping three-member panels of justices, with every member potentially having to participate in a ruling on the issue (in certain cases the entire court may also decide *en banc,* with members of the multiple previous panels joining in a single decision).[5] A dataset of court decisions emerging from this institution would be remarkably similar to the one proposed in this paper.

Degan and Merlo (2009) have explored empirical implications of sincere (versus strategic) voting. In fact, if the formal decision rule is known, this work may be reinterpreted precisely as the test of voter sincerity: if I know how the votes are counted, violations of the conditions established here could only be interpreted as indications that the scores do not directly reflect rational individual preference. Thus, to the extent one maintains the assumption that voters are rational, sincere voting would be falsified in this case. Other closely related work includes that of Kalandrakis (2010) on rationalizing voting decisions in a spatial setting and that by Apesteguia *et al.* (2014), who characterize observable outcomes of sincere and strategic application of various agenda rules employed in legislatures. In contrast with those papers, however, I do not assume observability of individual votes or preferences, but try to infer votes from observing the group choice data. Finally, it should be noted that the idea that committee membership variation may be used as a source of "revealed voting", as well as Example 3, the CARP, and a version of Proposition 3 in this paper, were previously presented in my earlier note (Gomberg 2011). With those exceptions, all formulations and results of this paper, however, postdate that note.

The rest of this paper is organized as follows. In section two I introduce the basic ideas, while characterizing choices that could be generated by a common voting rule (unanimity with default) in a binary choice setting. In section three I introduce the weighted majority rule and provide a characterization of restrictions they impose on committee choices In section four I extend the analysis to the case of three or more alternatives, while reinterpreting the weighted majority as scoring, and section five concludes.

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authority - of the *writ of amparo* in Mexican legal system.

[5]The details of the case assignment are described in Articles 49 and 50 of the *Reglamento Interno* (Internal Rules) of the Colombian Constitutional Court. The author thanks Juan Bertomeu for drawing his attention to this arrangement.

2 Unanimity with a Default

In order to fix the ideas, I shall start by proposing a simple characterization of choice patterns that may be generated by a fixed choice rule in a setting with just two alternatives. The rule I characterize is, in fact, quite common: unanimity with a default (this could, for instance, be a rule used in a criminal jury trial: unless all members of the jury vote to convict the defendant is declared innocent). Suppose we observe a collection of decisions of different committees with varying membership on the same binary issue and believe that all of these committees used this rule. Would we be able to test this theory using this data, without observing individual votes and without knowing, what the default choice is? In other words, are there restrictions on the committee choices such that if these are violated, the theory is conclusively falsified?

In order to answer this question, it shall be useful to define the notion of a *committee choice structure*. Let $X = \{a, b\}$ be the set of alternatives that committees face (it could be the Guilty versus Innocent, if the committee is a jury, or approve versus reject, if we are dealing with congressional committees considering nominations, etc.). Let a finite set $N = \{1, 2, ...n\}$ denote the general pool of agents, from which the committees are drawn and let $\mathcal{E} \subset 2^N \setminus \{\varnothing\}$ be the set of committees, that are observed and let the function $C : \mathcal{E} \to X$ denote the committee choice.[6] The *committee choice structure* is defined as a pair $(\mathcal{E}, C(.))$ - a record of observed committee choices. I shall view it as a data set, which may be used by an observer to deduce the preference profiles and the preference aggregation rules the committee uses.

I shall assume that each individual $i \in N$ has a well-defined complete preference relation over X, $\succcurlyeq^i$(its asymmetric part being denoted as $\succ^i$). One can define the committee choice rule to be unanimity with default $y \in X$ if $C(S) = y$ unless for $x \neq y$ we have $x \succ^i y$ for all $i \in S$ (in general, individuals' true preferences might not be directly observable, but in a binary choice environment under the unanimity with a default rule it is a weakly dominant strategy for all players to simply declare their true preference, irrespective of who else is in the committee).

Suppose that we observe that the committee choice structure satisfies the following two properties:

[6]The reason I am only considering choice functions, rather than choice correspondences in this section - an assumption that will be relaxed later - is that the unanimity rule, as here defined, always produces a unique choice.

$U1$ (expansion): for any $S, T \in \mathcal{E}$ such that $(S \cup T) \in \mathcal{E}$ and $C(S) = C(T) = x \in X$ it must be that $C(S \cup T) = x$.

$U2$ (no more than one reversal): for any $S, T, Q \in \mathcal{E}$ such that $S \subset T \subset Q$ and $C(S) \neq C(T)$ then $C(Q) = C(T)$.

It is pretty straightforward to see that the two properties here defined are necessary consequences of the unanimity decision-making with default. Indeed, if two committees both came to the same decision, then either it is the default (in which case at least one member of the joint committee must be in favor of sticking to it), or it isn't (in which case the joint committee must be unanimous in its desire to overturn the default). In both cases, the joint committee must agree with the decision of its subcommittees, satisfying expansion. Furthermore, if a larger committee disagrees with the smaller one, it must be the case that one of its members - and none of the members of the subcommittee - must be in favor of sticking to the default. Once the default is thus imposed, further inclusion of additional members can no longer affect committee choice.

A less immediate observation is that, if the set $\mathcal{E}$ is sufficiently rich (*i.e.*, if it is closed under unions), the above two properties completely characterize the committee choice structures over it that can be *rationalized* with this group decision rule. In this case, by *rationalizability* I mean existence of an actual preference profile $\succsim = \{\succsim^i\}_{i \in N}$ and a default alternative $x \in X$ such that application of the unanimity with a default would produce choices identical to those observed in the data. In fact, the following simple proposition can be easily proved:

Proposition 1 *Let $\mathcal{E}$ be closed under unions.*[7] *Then a committee choice structure over $\mathcal{E}$ is rationalizable by unanimity with default if and only if it satisfies $U1$ and $U2$.*

Proof: The "only if " part I have demonstrated above, so it remains to show that for every such committee choice structure I can find a profile of individual preferences that would generate the original choice structure. I shall do this by construction. If $C(S) = x \in X$ for every $S \in \mathcal{E}$, then we can obtain rationalization trivially.[8] Indeed, taking, without loss of generality $x = a$ we may define $a \succ^i b$ for all $i \in N$, which will immediately generate the

[7]That is, if $S, T \in \mathcal{E}$, then $S \cup T \in \mathcal{E}$.

[8]This of course, covers the case when $\mathcal{E}$ has a single element

original choice. Hence, the interesting case is when there exist $Q, W \in \mathcal{E}$ such that $C(Q) \neq C(W)$. Since $\mathcal{E}$ is closed under unions we may, without loss of generality take $C(Q \cup W) = C(Q) = a \neq b = C(W)$. For each $i \in S$ such that $C(S) = b$, I shall define $b \succ^i a$. I shall define $a \succ^i b$ for all the remaining i. Clearly, $\succ^i$ is an asymmetric binary relation on $X = \{a, b\}$, which can be completed to an anti-symmetric relation by assuming reflexivity. I will now show that it, indeed, generates the original committee choice structure, *i.e.* that $b = C(S)$ if and only $b \succ^i a$ for all $i \in S$, and $C(S) = a$ otherwise. Clearly, if $b = C(S), b \succ^i a$ follows by construction for all $i \in S$. Suppose $b \succ^i a$ for all $i \in S$, but $C(S) = a$. From the construction of $\succ^i$ we know that for each $i \in S$ there exists $S_i \in \mathcal{E}$ such that $C(S_i) = b$ and $i \in S_i$. Since $\mathcal{E}$ is closed under unions, by $U1$ we know that since $C(Q \cup W) = C(S) = a$, we must have $C(Q \cup W \cup S) = a$. Since $S \subset (S \cup W) \subset (Q \cup W \cup S)$ it follows by $U2$ that $C(S \cup W) = a$. But by $U1$ we have $C((\cup_{i \in S} S_i) \cup W) = b$, and $W \subset (S \cup W) \subset ((\cup_{i \in S} S_i) \cup W)$ so that, by $U2$, $C(S \cup W) = b$ - contradiction. Q.E.D.

Though simple, this proposition provides us with a characterization, that allows us to test a hypothesis that the groups were making their choices using a common voting rule - and, in fact, it allows us to determine from data what the default is even if we do not know it.[9] Of course, this assumes that each voter was taking a decision without regard to the presence or votes of the other committee members: an assumption plausible if we are dealing with preference aggregation in a legislature (in which case sincerely voting one's preference would be a weakly dominant strategy), but less likely if we are dealing with information aggregation in a jury setting, where strategic voting might be important.[10] Hence, the condition developed here could be used as a test of voting sincerity - and, consequently, of preference versus information aggregation.

The assumption that $\mathcal{E}$ is closed under unions appears restrictive, as it would fail in many realistic circumstances. Thus most *tutela* decisions in Colombia are adopted by 3-member court panels. Likewise, in many experimental papers analyzing group decision-making report data on decisions by

[9]It follows from the proof of the proposition above that, except in the trivial case where the choice is the same for all committees, the choice of the default is uniquely implied by the data.

[10]See, for instance, Feddersen and Pesendorfer (1996).

fixed-size overlapping committees.[11] Clearly, the fixed committee size would preclude $\mathcal{E}$ being closed under unions. Fortunately, this assumption can be easily relaxed. Even if a particular committee T is never observed in the data, if it can be decomposed into a union of elements of either $\mathcal{A}$ or $\mathcal{B}$, *expansion* would allow us to impute its decision. I shall define its closure $\mathcal{E}^* = \left\{S \in 2^N \backslash \{\varnothing\} : S = \cup_i S_i,\, S_i \in \mathcal{E}\right\}$.

For every choice structure $(\mathcal{E}, C(.))$ one may now define an $x-$*extension* $(\mathcal{E}^*, C_x^*(.))$

$$C_x^*(S) = \begin{cases} x, \text{if } S = \cup_i S_i \text{ for some } \{S_i\}_{i=1..m} \subset \mathcal{E} \text{ such that } C(S_j) = b \text{ for some } j \\ y \neq x, \text{ otherwise} \end{cases}$$

We can now formulate the following simple corollary to the previous proposition.

Corollary 1 *A committee choice structure over $\mathcal{E}$ is rationalizable by unanimity with default x if and only if $(\mathcal{E}^*, C_x^*(.))$ satisfies $U1$ and $U2$.*

Proof. Note that, as long as $U1$ holds, $C(S) = C_x^*(S)$ for any $S \in \mathcal{E}$. Hence, since $(\mathcal{E}^*, C_x^*(.))$ is rationalizable by proposition 1, so is $(\mathcal{E}, C(.))$. The necessity of $U1$ and $U2$ is likewise straightforward. Indeed, suppose $C_x^*(S) = x$ for some $S \in \mathcal{E}^*$ By construction there exists $\varnothing \neq S' \subset S$ such that $C(S') = x$. Hence, for any rationalizing preference profile $\succsim$ there must exist $j \in S' \subset S$ such that $x \succcurlyeq^j y$, which would imply, under the unanimity with default x that any coalition $T \in \mathcal{E}$ containing j must have $C(S) = x$, which, by construction, implies $C_x^*(T) = b$ for every $T \supset S$, from which $U2$ on C_x^* follows. As for the $U1$, it remains to show that for any $C_x^*(S) = C_x^*(T) = y \neq x$ implies $C_x^*(S \cup T) = y$. But, of course, if $C_x^*(S \cup T) = x$ then, by the same logic there must exist $j \in S \cup T$ such that $x \succcurlyeq^j y$. Without loss of generality let $j \in S = \cup_i S_i$, $S_i \in \mathcal{E}$ from which it follows that j belongs to some $S_k \in \mathcal{E}$, implying that $C(S_k) = C_x^*(S) = x$ - contradiction.

Example 1 *Consider $N = \{1, 2, 3, 4\}$. Suppose the following choices are observed: $C(\{1, 2\}) = C(\{2, 3\}) = C(\{3, 4\}) = b$; $C(\{1, 3\}) = C(1, 4) = a$. Though we do not observe choices of these committees, for any extension $U1$*

[11]For a recent survey of such studies see, for instance, Kugler *et al.* (2012). The committee overlap typically arises there from rematching committee members in order to avoid repeated interaction effects between experimental subjects.

implies $C_b^*(\{1,3,4\}) = a$ *and* $C_b^*(\{1,2,3,4\}) = b$*, which will, of course, imply violation of* $U2$*. Hence, this choice may not be rationalized by unanimity with default* b*.*

3 (Weighted) majority

In this section, as above, I will still consider the binary choice space, $X = \{a, b\}$. Suppose now that we conjecture that, instead of the unanimity with a default, the group is using some sort of a majority rule, in which each agent votes for his or her favorite alternative.

Consider, first, the simple majority rule. Under this rule, each agent submits a vote $v_i : X \to \{0, 1\}$. Note that this formulation allows for an abstention, which may be implemented by submitting the same vote for both alternatives. As in the previous section, I shall assume that the votes do not depend on who else is in the committee.

Given a vote from each of its members a committee S chooses an alternative that gets the highest score

$$C^m(S) = \arg\max_{x \in X} \sum_{i \in S} v_i(x)$$

where $\sum_{i \in S} v_i(x)$ is called the *score* received by an alternative $x \in X$ in voting by committee S. Such a choice structure is said to be generated by a simple majority rule. Note, that, since there could be voting ties, in this section I shall allow the choices to be multivalued, so that, in general, $C^m(S)$ would be a (non-empty-valued) correspondence. It may be noted (following Myerson 1995), that one could allow agents to submit votes that are distinct from reporting their preference orderings, whatever these may be. All the simple majority rule requires agents to do, is to report a ranking of alternatives in X by means of their votes $v_i \in \mathbb{R}^2$. Though this is not necessary, it may be convenient to view the votes as representing some underlying preference relations, so that $v_i(x) \geq v_i(y)$ if and only if $x \succsim^i y$.

If a committee choice structure $(\mathcal{E}, C(.))$, where $C : \mathcal{E} \rightrightarrows X$ is a non-empty-valued correspondence, is such that for any $S \in \mathcal{E}$

$$C(S) = C^m(S)$$

for some vote profile $v = \{v_i(.)\}_{i \in N}$ consistent with a preference profile $\succsim = \{\succsim^i\}_{i \in N}$ I shall say that $\succsim$ rationalizes $(\mathcal{E}, C(.))$ via a simple majority rule.

It is clear that not every such committee choice structure would be rationalizable with simple majority. Crucially, the notion of majority vote studied here implies that each individual's votes are independent of the committee composition: this may be interpreted as an implication of sincerity. Hence, if we ever observe that for two disjoint committees $S \cap T = \varnothing$ we have $C(S) = C(T)$ it must, indeed, follow that $C(S \cup T) = C(S)$. This property, introduced, for instance, in characterizations of scoring rules by Smith (1973) and Young (1975) is usually known as the *reinforcement* axiom and, at least in the single-valued choice case, is, clearly, implied by the *expansion* condition of the previous section. Clearly, reinforcement must be a necessary condition for the rationalizability here desired. But the simple majority has an even stronger implication for the actual scores that committees assign to alternatives: the score difference between the alternatives must be added up if two disjoint committees are joined.

In fact, if simple majority is the rule used, the difference w between the scores assigned to a and to b by the committee S

$$w(S) = \sum_{i \in S} v_i(a) - \sum_{i \in S} v_i(b)$$

will define a (signed) measure on the finite measurable space $(N, 2^N)$, as long as one naturally sets $w(\varnothing) = 0$, since $w(S \cup T) = w(S) + w(T) - w(S \cup T)$ for any two committees $S, T \in 2^N$.

However, we do not observe the actual scores or their differences, but only choices, which correspond to the sign of w. Defining $\bar{\mathcal{E}} = \mathcal{E} \cup \varnothing$ it may be convenient to summarize our observations with a function $f : \bar{\mathcal{E}} \to \{-1, 0, 1\}$ defined by the

$$f(S) = \begin{cases} -1, \text{ if } C(S) = \{b\} \\ 0, \text{if } C(S) = \{a, b\} \text{ or } S = \varnothing \\ 1, \text{if } C(S) = \{a\} \end{cases}$$

This function f is, of course, non-additive. If, however, we can, consistently with it, assign individual vote differences $w_j = w(\{j\})$to each individual in such a way that

$$sign(w(S)) = sign\left(\sum_{j \in S} w_j\right) = f(S)$$

we shall obtain a simple–majority-based theory that would explain how the observed choice structure arose!

Fortunately, it turns out that this problem is closely related to well-established problems in utility theory. In fact, a very similar mathematical problem emerges if one considers the question of when could a binary relation "at least as likely as" over a finite states space be represented by a probability measure, which has been posed and solved by Kraft et al. (1959). The following example, which is, essentially, a reinterpretation for the present setting of the one they constructed, implies that the reinforcement alone, though necessary, is not sufficient rationalizability by a simple majority rule.

Example 2 *Suppose* $N = \{1, 2, 3\}$, $f(\{1, 2, 3\}) = 1$ *and* $f(\{1, 2\}) = f(\{1, 3\}) = f(\{2, 3\}) = -1$. *The example does not violate reinforcement: there are no disjoint committees, taking the same decision. However, it is not hard to see that this set of choices is not consistent with simple majority. Indeed, since every committee of two members is taking the same decision it follows that* $2w_1 + 2w_2 + 2w_3 < 0$. *However, the grand coalition's decision implies that* $w_1 + w_2 + w_3 > 0$ *- an obvious contradiction.*

As noted above, *expansion* is a stronger condition than *reinforcement* and it, clearly, fails in the example above, which may tempt one to consider if it could the right condition for our axiomatization. However, it is too strong, in the sense that simple majority can easily generate examples that would violate expansion.

Example 3 *Suppose* $N = \{1, 2, 3, 4, 5, 6, 7\}$, $f(\{1, 2, 3, 4, 5, 6, 7\}) = 1$ *and* $f(\{1, 2, 3, 4, 5\}) = f(\{3, 4, 5, 6, 7\}) = -1$. *By taking the union of the last two coalitions we can easily see that expansion fails here. However, this could be easily generated by the simple majority rule if agents* $1, 2, 6,$ *and* 7 *consistently vote for* a *and the rest vote for* b.

Consequently, a different condition, which I shall call *strong reinforcement,* is required, which is analogous to the the *strong additivity* of Kraft et al. (1959). Following Fishburn (1986) it can be presented as follows. Consider two collections (of equal cardinality) of committees $\mathbf{S} = (S^1, S^2, ..., S^m)$ and $\mathbf{T} = (T^1, T^2, ..., T^m)$, sucht that $S^i, T^i \subset N$ for all i.. Note, that an empty set is taken here as a possible committee and that a committee might be repeated several times within a collection. Denote as $n_j(\mathbf{S})$ the number of committees in the collection $\mathbf{S}$ that individual j is included in. We say that $\mathbf{S} \approxeq \mathbf{T}$

if for each individual $j \in N$ $n_j(\mathbf{S}) = n_j(\mathbf{T})$. Thus, in the example 2 above, if we define $\mathbf{S} = \{\{1,2,3\},\{1,2,3\},\varnothing\}$ and $\mathbf{T} = \{\{1,2\},\{1,3\},\{2,3\}\}$ we will have $\mathbf{S} \approxeq \mathbf{T}$, since each individual is observed exactly twice in each committee collection. I am now ready to define the following condition of choice structures:

$W1$. The choice correspondence C satisfies *strong reinforcement* if for each pair of committee collections $\mathbf{S}, \mathbf{T}$ such that $\mathbf{S} \approxeq \mathbf{T}$ if $f(S^i) > f(T^i)$ or $f(S^i) = f(T^i) = 0$ for $i = 1, 2, ..., m-1$ then $f(S^m) \leq f(T^m)$.

It is obvious from Example 2 that strong reinforcement would have to hold if a choice structure is to be explainable with a simple majority vote. There would, of course, be additional conditions required for a simple majority characterization, which would arise from the *anonymity* axiom, intrinsic in the characterization of the simple majority (May 1952). However, if we allow for the voting weights to vary by agent, the strong reinforcement turns out to fully characterize the impact of majoritarianism on the choice structure of this sort!

Formally, the weighted majority voting may be introduced by defining the votes to be $v_i : X \to \{0, W_i\}$, where $W_i > 0$ stands for the voting weight of individual i (up to this point we have forced $W_i = 1$ for all agents). We shall assume that these weights are not observable by the outsider, who will now be trying to derive them from data. The rest of the model and the definitions remain unchanged. In particular, we may define

$$C^{wm}(S) = \arg\max_{x \in X} \sum_{i \in S} v_i(x)$$

We say that a committee choice structure $(\mathcal{E}, C(.))$, is rationalizable by a weighted majority rule if we can find a preference profile $\succsim$ and a collection of individual weights $(w_1, w_2, ...w_N) \in \mathbb{R}_+^N$ such that

$$C(S) = C^{wm}(S)$$

Theorem 1 *A committee choice structure* $(\mathcal{E}, C(.))$ *is rationalizable by a weighted majority rule if and only if the choice structure satisfies strong reinforcement.*

Proof. The necessity part is straightforward, since if it were not the case, there would exist a pair of committee collections $\mathbf{S} \approxeq \mathbf{T}$ such that

$f(S_i) > f(T_i)$ or $f(S_i) = f(T_i) = 0$ for all $i = 1, 2, ..., m-1$ and $f(S_m) > f(T_m)$ However, as $f(S_i) = sign(w(S_i)) = sign\left({}_{j\in S_i} w(\{j\})\right)$ it follows that ${}_{j\in S_i} w_j >_{j\in T_i} w_j$ or ${}_{j\in S_i} w_j =_{j\in T_i} w_j = 0$ for $i = 1, 2, ...m-1$ and ${}_{j\in S_m} w_j >_{j\in T_m} w_j$, which, if we some across the committees in each collection, in turn would imply that ${}_{j\in N} n_j(\mathbf{S})\, w_j >_{j\in N} n_j(\mathbf{T})\, w_j$ - a contradiction.

The proof of sufficiency closely follows that of Theorem 4.1 in Fishburn (1970). If all committees make the same choice, the theorem is trivially true, therefore, I shall henceforth assume that there exists at least one pair of committees $(S,T) \in \bar{\mathcal{E}}\times\bar{\mathcal{E}}$ such that $f(S) > f(T)$ Let $K \in \mathbb{N}$ be equal to the number of distinct committee pairs $(S,T) \in \bar{\mathcal{E}}\times\bar{\mathcal{E}}$ such that $f(S) > f(T)$ and $M \in \mathbb{Z}_+$ be equal to one half of the number of committee pairs $(S,T) \in \bar{\mathcal{E}}\times\bar{\mathcal{E}}$ such that $S \neq T$ and $f(S) = f(T) = 0$ (note that this includes committee pairs of the form $(S, \varnothing)$ and $(\varnothing, T)$). Clearly, $K + M \leq 2^{2n} < \infty$.

For each committee S let the indicator function

$$1_S(j) = \begin{cases} 1 \text{ if } j \in S \\ 0 \text{ if } j \notin S \end{cases}$$

For each of the first $k = 1, 2..., K$ committee pairs S^k, T^k defined above we may write

$$ {}_{j=1}^{n} w_j a_j^k > 0$$

and for each of the following $k = K+1, K+2, ..., K+M$ committee pairs S^k, T^k we may write

$$ {}_{j=1}^{n} w_j a_j^k = 0$$

where $a_j^k = (1_{S^k}(j) - 1_{T^k}(j)) \in \{-1, 0, 1\}$, and the weights ${}_{j=1}^{n} w_j$ may be interpreted as a "reconstruction" of the individual vote difference consistent with the observed choice structure (note, in particular, that this would imply that ${}_{j=1}^{n} w_j 1_S(j) = 0$ for every S such that $f(S) = 0$).

Suppose no such weights $w = (w_1, w_2, ..., w_n)$ may be found. Then by Theorem 4.2 in Fishburn (1970), known as the Theorem of the Alternative, there must exist a collection of numbers $r_{k,}$ $k = 1, 2, ..., M+K$, such that the first K of these are non-negative and not all zero so that for every $j = 1, 2, ..., n$

$$ {}_{k=1}^{K+M} r_k a_j^k = 0$$

In fact, since all a_j^k are rational by construction, all r_k may be chosen to be integers. If for some $k > K$ there is an $r_k < 0$ one may replace a_j^k with $-a_j^k$ to make it positive (this is possible since if $f\left(S^k\right) = f\left(T^k\right)$ one may interchange

S^k and T^k). Consider now two committee collections $\mathbf{S}$ and $\mathbf{T}$ such that each committee S^k is repeated r_k times in $\mathbf{S}$ and each committee T^k is repeated r_k times in $\mathbf{T}$. By construction the cardinality of each committee collection is equal to$_{k=1}^{K+M} r_k$ and from the preceding equation it follows that the number of times each individual is included in committees in each collection is

$$n_j(\mathbf{S}) =_{k=1}^{K+M} r_k 1_{S^k}(j) =_{k=1}^{K+M} r_k 1_{T^k}(j) = n_j(\mathbf{T})$$

and, hence $\mathbf{S} \approxeq \mathbf{T}$. But by construction we have $f\left(S^k\right) \geq f\left(T^k\right)$ for all $k = 1, 2, ...K + M$, with the first K inequalities strict. Hence, the strong reinforcement of the committee choice structure is violated. QED

Notably the proof of the theorem provides us not only with a way of falsifying the "weighted majority theory" of the committee decision-making, but also, assuming the strong reinforcement holds, with a way of calculating individual weights consistent with choices: except for the abstaining individuals, these would be the solutions w_j to the inequality system used in the proof.[12]

What if the system turns out not to have a solution? Of course, it is possible that some other group decision-making rule was used. Alternatively,

[12]Indeed, if we consider Example 2 above, we obtain the following system of $K = 7$ inequalities:

$$\begin{cases} w_1 + w_2 + w_3 > 0 \\ w_1 > 0 \\ w_2 > 0 \\ w_3 > 0 \\ -w_1 - w_2 > 0 \\ -w_1 - w_3 > 0 \\ -w_2 - w_3 > 0 \end{cases}$$

which, obviously, has no solution. The dual problem solves with $r_1 = r_2 = r_3 = r_4 = 3$ and $r_5 = r_6 = r_7 = 2$.

On the other hand, if the choice of the committee $\{2, 3\}$ were not observed the corresponding system would have been ($K = 5$):

$$\begin{cases} w_1 + w_2 + w_3 > 0 \\ w_2 > 0 \\ w_3 > 0 \\ -w_1 - w_2 > 0 \\ -w_1 - w_3 > 0 \end{cases}$$

which would be solved, for instance, by the weight vector $w = (-3, 2, 2)$.

we may suppose that some of the agents varied their vote based on committee membership - as could be the case, for instance, if they voted strategically. An important question that could be asked is, if we could try to identify the identity of these agents from data. Of course, if all agents did these, any data would be rationalizable. Hence, perhaps a more meaningful question could be, what is the *smallest* (by inclusion) set of agents that would *have to* behave strategically for the data to be rationalizable with a weighted majority vote.

4 Scoring and multiple alternatives

In the previous section I have assumed that the (not directly observed) weight v_i was an intrinsic characteristic of individual i. Alternatively, one may take a view that the weights arise from the manner in which a decision rule "converts" individual preferences into scores. Thus, for instance, in a simple (unweighted) plurality rule a score of 1 is assigned to the top choice of an individual and scores zero to all other alternatives, while under the Borda count the top alternative gets the maximal score n, the next best alternative gets a score of $n-1$, etc. The class of rules, such as plurality and Borda count, in which individuals are asked to provide each alternative with a numeric score (reflecting their preferences), the individual scores are added up and the alternative with the highest aggregate score is chosen is known as *scoring*. These rules have long been characterized by social choice theorists (see, for instance, Young 1975 or Myerson 1995) and are frequently used in practice. In this section I shall consider the problem of which committee choice structures could be rationalized by *some* scoring rule. This problem has previously been considered this problem in my earlier note (Gomberg 2011), though that note provides only a necessary consequence of decision-making by scoring, rather than a complete characterization of the committee choice structures rationalizable with scoring.

For the case of two alternatives, $X = \{a, b\}$, in fact, the problem is, essentially, identical to the weighted majority problem I considered in the previous section. However, this view provides a natural interpretation for the extension to the case of multiple alternatives, $X = \{x_1, x_2....x_n\}$, which I shall consider here.

The scoring rules require agents to report a ranking of alternatives in X by means of their votes $v_i : X \to \mathbb{R}$. As before, it is not necessary to assume that these actually come from an underlying preference, though, for interpretational reasons we shall still find it convenient to assume that $v_i(x) \geq v_i(y)$ if and only if $x \succcurlyeq^i y$. In the case of multiple alternatives, however, strategic voting incentives may arise even when the group's task is to aggregate preferences, rather than information. In this sense, the interpretation of the characterization that follows as providing a test for *sincere* voting, becomes most natural.

Similarly to the previous section, if a committee choice structure $(\mathcal{E}, C(.))$, where $C : \mathcal{E} \rightrightarrows X$ is a non-empty-valued correspondence, is such that for any $S \in \mathcal{E}$

$$C(S) = C^{scoring}(S) = \arg\max_{x \in X} \sum_{i \in S} v_i(x)$$

where the votes are consistent with preferences for some rational preference profile $\succsim$. I shall say that $\succsim$ rationalizes $(\mathcal{E}, C(.))$ via a scoring rule.

If there are three or more alternatives the problem cannot be reduced to that of an existence of a single measure on the committee space. Nevertheless, as long as all the committees are facing the same choice problem (i.e., the budget set B is not varied), the linear structure of the scoring rules utilized in the previous section allows for a very similar formulation.

Our basic objective remains the same: to find vote scores for each individual that would explain the observed committee choices. Notably, once there are at least three alternatives, we now will have to avoid "scoring cycles", as the following example shows.

Example 4 *(Gomberg 2011) Consider the alternative set $X = \{a, b, c\}$ and the four disjoint committees S_1, S_2, S_3 and T. Let $C(S_1) = a$, $C(S_2) = b, C(S_3) = c$, $C(S_1 \cup T) = b, C(S_2 \cup T) = c, C(S_3 \cup T) = a$. It is not hard to see that this implies that in committee T, b would have to gets higher score than c, c higher than a and a higher than b, which, of course, is an impossible cycle.*

As the example above suggests, the scores may be "revealed" through observed committee choices (the revelation idea first introduced in Gomberg 2011). For the rest of this section, I shall assume that there is a finite set of alternatives , $X = \{x_1, x_2, ...x_m\}$. The rest of the model is as before.

- *Direct revelation.* For each $S \in \mathcal{E}$ a pair of nested binary relations $P_S^* \subset R_S^*$ on X is defined by

  (i) let $x \in C(S)$ then xR_S^*y for any $y \in X$

  (ii) let $x \in C(S)$ and $y \notin C(S)$ for some $y \in X$ then xP_S^*y

This constitutes a record of direct preference revelation: if an alternative is chosen, it implies it received at least as high a score as any other feasible alternative and a strictly higher score than any feasible alternative not chosen.

Consider the total set of observations we have. If our theory is correct and this choice is rationalized with scoring, in the actual vote count each observation of xP_S^*y it must have been obtained from $\sum_{i\in S} v_i(x) > \sum_{i\in S} v_i(y)$ and each xR_S^*y from $\sum_{i\in S} v_i(x) \geq \sum_{i\in S} v_i(y)$. These are, of course, linear inequalities. In fact, the set of all "revealed scoring" statements must have been generated by a system of linear inequalities, which would have to hold simultaneously for the rationalization to be possible.

As the cardinality $\#X = m$, consider a vector $w = (w_1, w_2, ..w_n, w_{n+1}, ...w_{2n}, ...w_{nm}) \in R_+^{nm}$ where w_{kn+j} corresponds to the reconstructed vote that agent k emits for alternative j. As in the previous section, I shall consider each revealed scoring statement (taking care to track the committee by which it has been generated). As the total number of such statements is finite, let K be the number of strict statements xP_S^*y and M be one half of the rest.

Consider a list of all such revealed scoring pairs If the kth pair is $x_pP_S^*x_r$ (for the first K elements of the list) or $x_pR_S^*x_r$ (for the rest) then one can define $a_{jk} = 1$ for all $j = p + ms$, where $s \in S$, $a_{jk} = -1$ for all $j = r + ms$, where $s \in S$, and $a_{jk} = 0$ otherwise. The corresponding matrix $A = (a_{jk})$, which contains the relevant information about the observed choices, shall be called the *scoring matrix.*

As in the case of two alternatives, if for each of the first $k = 1, 2..., K$ revealed preference scoring relations defined above we may write

$$ {}_{j=1}^{n} w_j a_{jk} > 0 $$

and for each of the following $k = K + 1, K + 2, ...K + M$ revealed scoring relations we may write

$$ {}_{j=1}^{n} w_j a_{jk} = 0 $$

we would rationalize the observed choice structure.

As in the previous section, the Theorem of the Alternative allows one to restate the problem of existence of a solution to this system of inequalities as a problem of existence of a solution to the equation

$$ {}_{k=1}^{K+M} r_k a_{jk} = 0 \qquad (*) $$

where $(r_1, r_2, ...r_{K+M}) \in \mathbb{Z}^{K+M}$ with the first K terms non-negative and not all equal to zero. This, of course, constitutes the proof of the following proposition:

Proposition 2 *A committee choice structure* $(\mathcal{E}, C(.))$ *is rationalizable by a scoring rule if and only if the equation* $Ar = 0$ *(where* A *is the associated scoring matrix) has a non-zero integer solution* r*, with the first* K *coordinates of* r *all non-negative.*

As in the case of two alternatives, this condition is, in fact, necessary and sufficient for the existence of rationalization by scoring, though it is harder to get its intuitive interpretation. A greater feeling for its implication may be obtained if we reformulate a necessary implication of it in a more familiar "revealed preference" form (as earlier proposed in Gomberg 2011).

Consider, for instance, the "indirect revealed scoring" implied by the reinforcement property of the scoring rules (which, as noted above states that if two disjoint committees make the same choice from a given budget set, so should their union). We can then define the following .

- *Reinforcement*[13]

  The binary relations $P_S \subset R_S$ on X are defined by

  (i) xP^*y implies xPy, xR^*y implies xRy,

  (ii) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \cap T = \varnothing$, xR_Sy and xR_Ty imply that $xR_{S \cup T}y$

  (iii) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \cap T = \varnothing$, xP_Sy and xR_Ty imply that $xP_{S \cup T}y$

  (iv) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \subset T(T \backslash S \neq \varnothing)$, xP_Sy and yR_Tx imply that $yP_{T \backslash S}x$

  (v) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \subset T(T \backslash S \neq \varnothing)$, xR_Sy and yP_Tx imply that $yP_{T \backslash S}x$

With this in mind we may now define a simple acyclicity condition, motivated by the example above:

[13] Note that the example above shows that a stronger indirect extension could be imposed here. However, reinforcement is more intuitive, so I stick to it as a necessary implication of rationalizability.

$S1$ (Committee Axiom of Revealed Preference, CARP)[14]: For any $S \in 2^N \setminus \{\varnothing\}$ and any $x_1, x_2, ...x_n \in X, x_1 R_S x_2, x_2 R_S x_3 ... x_{n-1} R_S x_n$ implies $\neg (x_n P_S x_1)$

It is straightforward to see that CARP is, in fact, implied by scoring (its violation would imply a committee S assigning to an alternative x_1 both a higher and a lower score then to an alternative x_n, which is impossible). Hence, we have the following proposition:

Proposition 3 *A committee choice structure $(\mathcal{E}, C(.))$ is rationalizable by a scoring rule only if the implied R_S and P_S satisfy CARP for each $S \in 2^N \setminus \{\varnothing\}$.*

5 Conclusions and further research

The objective of this paper is to explore how observations of group actions may used to test theories about within-group decision procedures. For this purpose, I introduce the notion of a committee choice structure and establish a necessary and sufficient condition for such a choice structure to be rationalizable via two common voting rules: unanimity with a default and scoring, when the committees decide over the fixed budget set, with a natural interpretation for the case of two alternatives.

A key insight of the paper is that observations of collective decisions by themselves might be revealing, as long as we may observe variations on group membership. In particular, when the formal decision-making rule is known, the data-theoretic characterizations developed here may be used to develop tests for voting sincerity based purely on the outcomes of group decisions.[15] Indeed, a violation of the characterizing conditions under the observable group decision rule would indicate that individuals change their votes based on the group composition: as would be the case if they voted strategically. In a sense, such tests would therefore involve identifying from the data (groups of) agents that would have to behave strategically for the data to be rationalizable.

[14] The naming suggestion for this axiom, originally introduced in Gomberg (2011), belongs to Norman Schofield.

[15] As noted above, in the binary choice settings of sections 2 and 3, if individuals are merely aggregating preferences they would have no incentives for strategic voting. If, however, they share an underlying preference but aggregate information, strategic voting incentives would emerge, as in Feddersen and Pesendorfer (1996).

As mentioned in the introduction, data sets resembling those suggested in this paper might emerge from a corpus of court decisions, especially those from legal systems in which prior decisions do not constitute legal precedent. Compiling and analyzing such data set for insights on the preferences of court members remains to be done.

References

[1] Apesteguia, J., M. Ballester, and Y. Masatlioglu (2014), A Foundation for Strategic Agenda Voting, *Games and Economic Behavior,* **87**: 91-99

[2] Corte Constitucional de Colombia (1992), *Reglamento Interno*, retrieved at http://www.corteconstitucional.gov.co/lacorte/reglamento.php

[3] Deb, R (1976), On Constructing Generalized Voting Paradoxes, Review of Economic Studies 43: 347-351

[4] Degan, A. and A. Merlo (2009), Do Voters Vote Ideologically, *Journal of Economic Theory* **144:** 1868-1894

[5] Feddersen, T. ,and W. Pesendorfer (1996), The swing voter's curse, *American Economic Review*, **86**: 408-424

[6] Fishburn, P. (1970), *Utility Theory for Decision-Making*, John Wiley and Sons : 1970

[7] Fisburn, P. (1986), Axioms of Subjective Probability, *Statistical Science* **1**: 335-345

[8] Gersbach, H. and V. Hahn (2008), Should the individual voting records of central bankers be published? *Social Choice and Welfare* **30**: 655-683

[9] Gomberg, A (2011), Vote Revelation: Empirical Content of Scoring Rules, in Schofield N., Caballero G., *Political Economy of Democracy and Voting*, Springer Verlag: 2011: 411-417

[10] Houthakker, H. (1950), Revealed Preference and the Utility Function, *Economica* **17**:159-174

[11] Kalandrakis, T. (2010), Rationalizable Voting, *Theoretical Economics* **5**: 93-125

[12] Kraft, C., J. Pratt. and A. Seidenberg (1959), Intuitive Probability on Finite Sets, *Annals of Statistics* **30**: 408-419

[13] Kugler, T., Kausel, E., and Kocher, M. (2012), Are groups more rational than individuals? A review of interactive decision making in groups. *Wiley Interdisciplinary Reviews: Cognitive Science*, **3**: 471-482.

[14] McGarvey, D. (1953), A Theorem on the Construction of Voting Paradoxes, *Econometrica* **21**: 608-610

[15] May, Kenneth O. (1952), A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica* **20**: 680-684.

[16] Myerson, R. (1995), Axiomatic Derivation of the Scoring Rules without the Ordering Assumption, *Social Choice and Welfare* **12**: 59-74

[17] Nakamura, K., (1979),The Vetoers in a Simple Game with Ordinal Preferences, *International Journal of Game Theory* **5**: 55-61

[18] Sibert, A. (2003), Monetary Policy Committees: Individual and Collective Reputations, *Review of Economic Studies* **70**: 649-665

[19] Smith, J. (1973), Aggregation of Preferences with Variable Electorate, *Econometrica* **41**: 1027-1041

[20] Vargas, J. (1996). Rebirth of the Supreme Court of Mexico: An Appraisal of President Zedillo's Judicial Reform of 1995, *The American University Journal of International Law and Policy* **11**: 295-341

[21] Young, P (1975), Social Choice Scoring Functions", *SIAM Journal on Applied Mathematics* **28**: 824 - 838