How many sorting equilibria are there (generically)?*

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Abstract

It is shown that in a generic two-jurisdiction model of the type introduced by Caplin and Nalebuff (1997), the number of sorting equilibria (with jurisdictions providing distinct policies) is finite and even.

1 Introduction

Caplin and Nalebuff (1997) introduce a general class of models with multiple jurisdictions, such as arise, for instance, in modeling local public goods or political parties, that can be used to generate equilibrium population sorting under free mobility. They also provide a sorting equilibrium existence result when there are two jurisdictions, but only for the case when jurisdictional policies are even-dimensional. Gomberg (2004) provides a similar result for the odd-dimensional case.

Until now, the issue of local uniqueness and the number of sorting equilibria in these models has remained unexplored. At the same time, much of the applied work in this area involves comparative statics analysis, which, implicitly, assumes, at least, local uniqueness of equilibrium. The present paper intends to fill in this gap in the literature.

The key difficulty involved in establishing both the existence results of the earlier work and those of this paper lies in the fact that, in addition to sorting equilibria, these models also, typically, possess equilibria in which all jurisdictions provide identical policies and the, consequently indifferent, population is distributed to support such policy outcomes. Unfortunately, this means that

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usual (i.e., Brouwer-like) fixed point results are not suitable to establish existence of *sorting* equilibria. Fortunately, in the two-jurisdiction case the problem may be reduced to a simple mathematical issue of existence of fixed points of deformations on spheres. An interesting implication of this is that sorting equilibria in models of this type are almost never unique when they exist; in fact, the number of such equilibria is, generically, even. The evenness part of the result becomes easier to understand if one recalls that I only consider the number of *one type* of equilibria - that is the number of equilibria with distinct jurisdictional policies and fully sorting population partitions.

One lesson of this paper is that index theory has a potential for establishing results on the existence and number of different types of equilibria (in this case, this would be the sorting equilibria). This suggests, that similar techniques could be useful in other models with distinct equilibrium types, such as, for instance, models with asymmetric information. In that case it could be possible to achieve non-constructive results for existence and number of pooling and separating equilibria. This is, indeed, related to the approach of Gale (1992) in his study of existence of separating equilibria in markets with adverse selection. Likewise, Chakraborty and Harbaugh (2007) employ related techniques in their work on multi-dimensional cheap talk games.

Aside from the difficulties arising from the fact that the underlying space is non-contractible, the mathematical tools employed in this paper have been standard in economics since Debreu (1970). Most of them can be found in Dierker (1974). For references on the Lefschetz fixed point Theorem see McLennan (1989) and Brown (1974).

The rest of the paper is organized as follows. Section 2 introduces the model, section 3 formulates the main results and section 4 discusses their intuition and indicates possible extensions.

2 The model

I shall first outline the model originally introduced by Caplin and Nalebuff (1997).

Let $M = \{1, 2\}$ be the set of jurisdictions;

$\emptyset \neq X_j \subset \mathbb{R}^n$ - the set of policies available to community $j = 1, 2$; for simplicity, I shall assume that X_j is bounded and *open* (the former assumption is relaxable).

$X \equiv X_1 \times X_2 \subset \mathbb{R}^{2n}$ - the set of all possible policy profiles;

$\mathcal{A} = (A, \mathcal{B}, \mu)$ - the measure space of agents' types, where $A \subset \mathbb{R}^n$, $\mathcal{B}$ is the (Borel) σ-algebra over A, μ is a non-atomic probability measure on A,

For each $\alpha \in A$ there is a continuos utility function $u_\alpha : X_1 \cup X_2 \rightarrow \mathbb{R}$, representing his/her preferences over the jurisdictional policies.

Assume that each individual must choose to join exactly one jurisdiction, resulting in a population partition.

Definition 1 *A pair of measures $\sigma = \{\sigma_1, \sigma_2\}$ over $(I, \mathcal{B})$ is called an admissible **population partition** if for any $C \in \mathcal{B}$ $\mu(C) = \sigma_1(C) + \sigma_2(C)$*

Given a population partition σ, I shall denote the support of jurisdiction j population measure σ_j as S_j. The set of all admissible population partitions shall be denoted as $\hat{\Sigma}$.

Every jurisdiction is assumed to have a well-defined policy rule $P_j : \hat{\Sigma} \to X_j$, which shall be called its **constitution** (in the terminology of Caplin and Nalebuff (1997) such constitutions are called *membership-based);* let $P \equiv \{P_1, P_2\}$ denote the profile of constitutions.

A membership-based multi-community economy is then defined as the list

$$E \equiv \left(M, X, P, \mathcal{A}, \{u_\alpha(.)\}_{\alpha \in A}\right)$$

An equilibrium is a population partition/policy profile such that everyone resides in the jurisdiction he or she prefers and policies are set according to constitutions. I shall concentrate on sorting equilibria, in which jurisdictions provide distinct policies and individuals of different types separate:

Definition 2 *A policy profile - admissible population partition pair* $(x, \sigma) \in X \times \hat{\Sigma}$ *is said to be a* ***sorting equilibrium*** *(thereafter, just* ***equilibrium****) of* E *if:*

(i) $S_j \subset \{\alpha \in A : u_\alpha(x_j) \geq u_\alpha(x_k),\ k \neq j\}$ *for every* $j = 1, 2$; *(free mobility);*
(ii) $P_j(S) = x_j$, *for every* $j = 1, 2$; *(constitutionality);*
(iii) $x_1 \neq x_2$ *(distinct policies).*

I impose a number of assumptions on agent preferences and jurisdictional constitutions, mostly following Caplin and Nalebuff (1997) and Gomberg (2004). The first assumption, introduced mainly to simplify the analysis, restricts the preferences to be linear:

Assumption 1: (linear preferences and bounded derivatives) an individual type $\alpha \subset A$ in jurisdiction j with the policy vector $x_j \in X_j$ receives utility

$$u_\alpha(x_j) = \alpha \cdot x_j + t(x_j)$$

where $t(x_j)$ is a bounded continuously differentiable function of x_j with bounded derivatives. Furthermore, $\nabla t(x_j) \ll 0$ for all $x_j \in X_j$.[1]

The second assumption ensures that the agents' distribution does not have a concentration along any hyperplane and that whatever policy profile faced by the agents, somebody would always prefer to join each jurisdiction (this second part of the assumption could be easily relaxed).

[1] The important requirement on the derivatives here is that they are non-zero, and it may be relaxed except if $n = 1$. In the latter case, if, for instance, $t(x) = const$ there are only two possible population partitions, and any discussion of genericity is vacuous (that there are two equilibria in this case is trivial). More generally, if when $n = 1$ it may occur that $t'(x) = 0$, one may not have enough ability to "perturb" the economy for the results to hold. For $n > 1$ the restriction is not essential - in fact, I do not use it at all in case n is even, and only use that $\frac{\partial t}{\partial x_{1j}} < 0$ for the case when n is odd. For the case of odd n and $t(x) = const$ a proof may be found in a working paper version of this paper, Gomberg (2003). The actual sign of the derivatives is less important: if derivatives were positive, the proofs would go through almost without changes.

Assumption 2 (no concentration of agents and full support):

(i)μ is hyperdiffuse[2] over A .

(ii) for any $(x_1, x_2) \in X$ there exist α, α' both in the support of μ such that $\alpha \cdot x_1 > \alpha \cdot x_2$ and $\alpha' \cdot x_1 < \alpha' \cdot x_2$

Under assumptions 1 and 2, given a policy profile such that $x_1 \neq x_2$, the set of indifferent agents will be a (zero-measure) hyperplane, while on each side of the hyperplane there will be a positive mass of agents strictly preferring one jurisdiction's policy over another. Therefore, if I identify together all population partitions that differ only by the jurisdiction choice of a zero measure of agents, a partition σ may be identified with the partition of the type space between the jurisdictions $S = (S_{1,} S_2)$, where S_j, as before, is the support of each jurisdiction's population measure. Consider a pair of distinct policies $x_1 \neq x_2$. The set of agents indifferent between these policies is a hyperplane given by the equation

$$\alpha \cdot \frac{(x_1 - x_2)}{||x_1 - x_2||} + \frac{t(x_1) - t(x_2)}{||x_1 - x_2||} = 0$$

with agents on one side of the hyperplane constituting the membership support of a jurisdiction. Clearly, space of all population partitions that can obtain form a policy profile such that $x_1 \neq x_2$ can be embedded into a cylinder $\mathbb{S}^{n-1} \times \mathbb{R} = \left\{(\kappa, b) \in \mathbb{R}^{n+1} : ||\kappa|| = 1, b \in \mathbb{R}\right\}$ where $\kappa = \frac{(x_1 - x_2)}{||x_1 - x_2||}$ is a unit vector orthogonal to the indifference hyperplane and $b = -\frac{t(x_1) - t(x_2)}{||x_1 - x_2||}$ is the intercept. Of course, not all points on the cylinder may be reachable as possible partitions - what may be possible would depend on the functional form of $t(x)$. However, it will, in fact be useful to consider a still larger set of partitions. In fact, I shall consider including the partitions with $b = \pm\infty$. Clearly, if $b = \pm\infty$ the entire population will be joining a single jurisdiction, irrespective of κ, which provides for natural "gluing" at the edges of the cylinder (we could, in fact, identify all the partitions with $|b|$ sufficiently large, but this is not necessary for this paper).I shall denote as e_1 $(b = -\infty)$ and e_2 $(b = \infty)$ the points such that the entire population is, respectively, in jurisdictions 1 and 2, and consider the set $\Sigma = \left(\mathbb{S}^{n-1} \times \mathbb{R} \cup \{e_1, e_2\}\right) \subset \hat{\Sigma}$. To summarize, every partition $S \in \Sigma$ may be parametrized by a pair $(\kappa, b) \in \mathbb{S}^{n-1} \times \mathbb{R}$ with the identification that all points $(\kappa, -\infty) = e_1$ and $(\kappa, \infty) = e_2$.

Assuming constitutions P_j disregard movements of zero-measure coalitions of agents, P is well-defined when viewed as a function on Σ. I furthermore assume that, given a partition with populations on the opposite sides of a hyperplane, the jurisdictions indeed choose different policies:

Assumption 3: (distinct policy outcomes) for any $S \in \Sigma$, $P_1(S) \neq P_2(S)$.

It remains to define a topology on the space Σ. Fortunately, it turns out that, given assumption 2 simply taking the usual Euclidean topology on the

[2] I.e., no hyperplane in A of dimension less then n may contain a positive measure of agents

parameter space $\mathbb{S}^{n-1} \times \mathbb{R}$ is rather natural, since it has the virtue of defining as "nearby" partitions that differ in actions of a small measure of agents. To deal with the continuity at e_1 and e_2 I shall, as is normally done (see, Munkres 1975, p.183), extend this topology to the entire Σ by defining the subset on Σ if it is either an open subset of $\mathbb{S}^{n-1} \times \mathbb{R}$ or a complement in Σ of a compact subset of $\mathbb{S}^{n-1} \times \mathbb{R}$ (I'd need to do this twice, first including e_1 and then e_2) This is the standard two-point compactification of the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$, which is well-known to be homeomorphic to a sphere $\mathbb{S}^n$. With this topology continuity of constitutions simply means that small population movements result in small policy changes (note, that this is the same strengthening of the Caplin and Nalebuff 1997 continuity that has been employed by Gomberg 2004).

Assumption 4*: (continuity) P is continuous on Σ.

Given a partition $S \in \Sigma$, the profile of policy functions P induces a position profile $x \in X$ ($x_1 \neq x_2$ by assumption 3) which, in turn, induces a new partition S'. Thus, one can define the **mobility map** $\phi : \Sigma \to \Sigma$ from the space of possible partitions to itself, given by

$$\phi(S) = (\phi_1(S), \phi_2(S)) = \left(\frac{P_1(S) - P_2(S)}{||P_1(S) - P_2(S)||}, -\frac{t(P_1(S)) - t(P_2(S))}{||P_1(S) - P_2(S)||} \right)$$

By construction, $\phi(S)$ is, indeed, in Σ. Furthermore, if assumption 4 holds, this map is continuous (note, that compactness of the domain and the continuity of P imply that $||P_1(S) - P_2(S)||$ is bounded away from zero). It is not hard to see that if S belongs to the set of fixed points of ϕ, $\mathcal{F}(\phi)$, the constitutional policy profile $P(S)$ induces back the population partition under free mobility S^*, i.e. $(S^*, P(S))$ is an equilibrium. Conversely, if $S \notin \mathcal{F}(\phi)$, S cannot be an equilibrium partition.

Proposition 3 *Under assumptions 1-3 the pair* $(x^*, S^*) \in X \times \Sigma$ *is a sorting equilibrium if and only if* $S^* \in \mathcal{F}(\phi)$ *and* $x^* = P(S^*)$.

Conditions 1-3 and 4* are sufficient for the major results of this paper (see the earlier version of this paper Gomberg 2003). Unfortunately, for many common models, the continuity assumption 4* may be too restrictive (in fact, sometimes one cannot even define a natural continuous extension of P to e_j). Fortunately, certain kinds of discontinuities can be dealt with. In fact, Caplin and Nalebuff (1997) have introduced a sufficient condition that works in the absence of continuity. I provide a slightly strengthened version of this assumption here:

Assumption 4: (continuity away from extinction and viability of small jurisdictions[3])

(i) P is continuous on $\Sigma \setminus \{e_1, e_2\}$

(ii) for each $j = 1, 2$ there exists an open neighborhood $V_j \subset \Sigma$ of e_j such that for any $S \in V_j \setminus e_j$ $\phi(S) \notin V_j$ and is bounded away from e_i $(i \neq j)$

[3] A complementary assumption of "contraction of small jurisdictions" would, likewise, work.

(iii) there exists a number $\eta > 0$ such that, for any $S \in V_j$, $||P(S_1) - P(S_2|| > \eta$.

Clearly, under assumption 4 no equilibrium may obtain in any of the V_j.

The final assumption used in Caplin and Nalebuff (1997) and Gomberg (2004), which can often be motivated by the jurisdictions choosing policies optimally, given their own populations, says that once the policies are set in a constitutional fashion, it will never be the case that the populations of towns simply reverse. In application to this paper it can be stated as:

Assumption 5 (weak Pareto condition)**:** for every $S \in \Sigma$ $\phi_1(S) \neq -\kappa$.

Since under assumption 4 the mapping $\phi(S)$ might not be continuous (or even well-defined) at e_j we may need an additional construction. Thus, for constitutions satisfying only 4 and not 4* we may define a new mapping as follows. Let $S = (\kappa, b)$ and define $b_2(\kappa) = \inf\{b' \in \mathbb{R} : (\kappa, b') \in V_2\}$, $b_1(\kappa) = \sup\{b' \in \mathbb{R} : (\kappa, b') \in V_1\}$ (the $b_j(\kappa)$ effectively trace out the boundaries of V_j). Now, let

$$\widehat{\phi}(S) = \left(\widehat{\phi}_1(S), \widehat{\phi}_2(S)\right) = \begin{cases} \phi(S), S \notin V_j \\ (\phi_1(S), \phi_2(S) + b - b_j(\kappa)), \text{ if } S \in V_j \\ e_j, \text{ if } S = e_j \end{cases}$$

The fixed points of the mapping $\widehat{\phi}$, other then the e_j (which are both fixed points by construction; however, since there are two of these, they do not affect the parity of the number of fixed points) coincide with those of ϕ. The continuity of $\widehat{\phi}(S)$ away from e under assumptions 4 and 5 is straightforward. Continuity at e_j follows from the fact that, under assumption 5, for any x, $(\phi_2(S) + b - b_j(\kappa)) \to \pm\infty$ as $b \to \pm\infty$.

2.1 Existence results

Caplin and Nalebuff (1997) and Gomberg (2004) provide a pair of sorting equilibrium results in this model. Both results are based on the observation that under assumptions 1-5 the mapping $\widehat{\phi}$ (or ϕ, if assumption 4 holds) is homotopic to the identity (*i.e.*, it is a deformation). When n is odd this immediately implies equilibrium existence, since any deformation on even-dimensional spheres must have fixed points (Gomberg 2004). This well-known result is closely related to the famous in the theory of dynamic systems "hairy ball theorem", and derives from the non-zero Euler characteristic of such spheres. For details of the proof of both existence results see Gomberg (2004).

As an aside one should mention that it does not have to be too surprising that sorting equilibrium existence results depend on the dimensionality n of the commodity space. In a sense that shall be further discussed below, the results are related to the properties of "out-of-equilibrium dynamics" implied by the mobility mapping $\widehat{\phi}$. And, of course, dimension-sensitive results have been well-known in the study of formal dynamic properties of economic models. Thus, for instance, in the general equilibrium setting, a unique equilibrium can be completely unstable only if the commodity space is *odd*-dimensional (Dierker 1974).

3 The number of equilibria

In this section it shall be shown that for "most" differentiable constitution profiles of two-jurisdiction economies there exists, at most, a finite number of equilibria. Furthermore, somewhat unusually, this number is guaranteed to be even.

The basic approach of this section is to define the class of "regular" smooth policy profiles for which the number of equilibria can easily be seen to be finite and then to show that this class, in fact, contains "almost all" smooth policy profiles.

I shall denote as $K \subset C^1(\Sigma \backslash \{e_1,e_2\}, X)$ the set of all continuously differentiable constitution profiles such that assumptions 3 - 5 hold.

Assumption 6: $P \in K$

It can be shown that if assumption 6 holds $\widehat{\phi} \in C^1(\Sigma \backslash \{e_1,e_2\}, \Sigma \backslash \{e_1,e_2\})$.

For any $f \in C^1$, $Df(x)$ denotes the derivative of f at x.

Definition 4 *An equilibrium* $(S^*, P(S^*))$ *in a differentiable two-jurisdiction model is called **regular** if the matrix* $\left(Id_{\mathbb{R}^n} - D\widehat{\phi}(S^*)\right)$ *is non-singular.*

For regular fixed points $S \in \mathcal{F}(\phi)$, the index of the fixed point $ind_S(\phi) = sgn|Id_{\mathbb{R}^n} - D\widehat{\phi}(S)| = \pm 1$. Though defining the differentiability at e_j might be tricky, there exists a generalization of the notion of the index that does not require it (see McLennan 1989). The "viability of small jurisdictions" part of assumption 4 is, in fact, sufficient to ensure that the both e_j are isolated fixed points that have the index equal to ± 1. Thus, for the purposes of what follows, they may be taken to be *regular*. It is immediate that any regular fixed point of $\widehat{\phi}$, other than e_j, corresponds to an *isolated* equilibrium of the model that is also robust to small perturbations of ϕ.

Definition 5 *A two-jurisdiction economy* $E = \left(M, X, P, \mathcal{A}, \{u_\alpha(.)\}_{\alpha \in A}\right)$ *is called **regular** if all fixed points of the mobility map* $\widehat{\phi}$ *are regular.*

It can be shown that regular economies are, in the usual sense, generic:

Proposition 6 *Under assumptions 1-6 the set* $\mathcal{R} \subset K$ *of constitutions* P *such that the economy is regular is open (in the topology of uniform convergence) and dense in* K.

Proof. *Part I. Density.*

I shall consider two separate cases: n *- even and* n *- odd.*

Case I: n *- even.*

Take an arbitrary $P \in K$. To show that the set of regular P is dense I shall "rotate" the mapping $\widehat{\phi}$ around the axis formed by the e_j.For every θ in a small neighborhood of $\frac{\pi}{2}$ consider the $n \times n$ square matrix

$$B_n(\theta) = \begin{bmatrix} \sin\theta & \cos\theta & 0 & 0 & ... & ... & 0 & 0 \\ -\cos\theta & \sin\theta & 0 & 0 & ... & ... & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta & ... & ... & 0 & 0 \\ 0 & 0 & -\cos\theta & \sin\theta & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... & \sin\theta & \cos\theta \\ 0 & 0 & 0 & 0 & ... & ... & -\cos\theta & \sin\theta \end{bmatrix}$$

Let

$$P_1(S;\theta) = B_n(\theta)\, P_1(S) + (I - B_n(\theta))\, P_2(S).$$

Notice that $P_1(S;\frac{\pi}{2}) = P_1(S)$ and for θ sufficiently close to $\frac{\pi}{2}$, $P_1(S;\theta)$ is close to $P_1(S)$. Therefore, since I have assumed that X_1 is open, $P_1(S;\theta)$ is a well-defined constitution for such θ.

Consider the constitution profile

$$P(S;\theta) = (P_1(S;\theta), P_2(S))$$

It can be easily shown that, one can approximate $P(S)$ by $P(S,\theta)$ for θ sufficiently close to $\frac{\pi}{2}$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\theta - \frac{\pi}{2}| < \delta$ implies $||P(S;\theta) - P(S)||_{C^1} < \varepsilon$ (where for $f \in C^1(X,Y)$, $||f||_{C^1} = \sup_{x\in X}|f(x)| + \sup_{x\in X;h=1,2;k=1,2...n} |\frac{\partial f_h}{\partial x_k}(x)|$).

Assumption 3 is sufficient to ensure that $P_1(S;\theta) \neq P_2(S)$ for θ sufficiently close to $\frac{\pi}{2}$ Therefore, the corresponding mobility map $\widehat{\phi}$ is well-defined. Indeed, its first coordinate has an easy expression

$$\widehat{\phi}_1(S;\theta) = B_n(\theta)\, \widehat{\phi}_1(S)$$

and, at least away from e_j the entire function $\widehat{\phi}$ is continuously differentiable in S and θ.

One can use a small neighborhood U of $\frac{\pi}{2}$ as a parameter space for the class of mobility functions. For any $S \in \Sigma \backslash \{e_1, e_2\}$, $\widehat{\phi}(.;\theta)$ is a submersion of $U \times \{S\}$ into $\mathbb{S}^n$ and therefore $\left(\widehat{\phi}(S;\theta) - S\right)$ is transversal to zero. Applying the transversality theorem (theorem 9.1 in Dierker 1974) and noting that the "poles" e_1,e_2 may be shown to be regular fixed points by direct computation (for instance, along the boundary of V_j), one gets that the set of θ with only regular fixed points is dense in U and that therefore the set of constitutions with only regular equilibria is dense in K.

Case II: n - odd.

Here I shall have to use a two-dimensional parameter family, simultaneously "rotating" and "stretching" the mapping $\widehat{\phi}$

Consider a scalar $\lambda \in \mathbb{R}$ *and the* $n \times n$ *matrix*

$$B_n(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & ... & ... & 0 & 0 \\ 0 & \sin\theta & \cos\theta & 0 & 0 & ... & ... & 0 & 0 \\ 0 & -\cos\theta & \sin\theta & 0 & 0 & ... & ... & 0 & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & ... & ... & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta & \sin\theta & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & ... & ... & \sin\theta & \cos\theta \\ 0 & 0 & 0 & 0 & 0 & ... & ... & -\cos\theta & \sin\theta \end{bmatrix}$$

Let

$$P_1(S;\theta,\lambda) = \lambda B_n(\theta) P_1(S) + (I - \lambda B_n(\theta)) P_2(S).$$

Thus, $P(S)$ *may be now approximated by*

$$P(S;\theta,\lambda) = (P_1(S,\theta,\lambda), P_2(S))$$

Consider the corresponding mobility mapping $\widehat{\phi}(S,\theta,\lambda)$*. It is well-defined in a sufficiently small neighborhood* U *of the point* $(\theta,\lambda) = \left(\frac{\pi}{2}, 1\right)$*. As before,*

$$\widehat{\phi}_1(S;\theta) = B_n(\theta)\widehat{\phi}_1(S)$$

However, in this case, this rotation does not move points with the first partition coordinates $\kappa = \pm(1,0,....0)$*, so using just* θ *as the local parametrization would not be sufficient to ensure local transversality in case any of these happened to be fixed points. However, at these points, as long as* $\frac{\partial t}{\partial x_{1j}} < 0$*, the* λ *has the effect of "stretching" the second coordinate* $\widehat{\phi}_2$*, ensuring that for any* $S \in \Sigma \backslash \{e_1, e_2\}$*,* $\phi(S;\theta,\lambda)$ *is a submersion of* U *into* $\mathbb{S}^n$ *and the rest of the argument above holds.*

Part II. $\mathcal{R}$ *is open.*

The set of all $\widehat{\phi} \in C^1(\Sigma \backslash \{e_1, e_2\}, \Sigma \backslash \{e_1, e_2\})$ *which have only regular fixed points is open in the* C^1 *topology of uniform convergence. Indeed, suppose otherwise. Then there exists a sequence* $\phi_k \to \widehat{\phi} \in \mathcal{R}$ *in* C^1*, such that for every* k $\det(Id_{\mathbb{S}^n} - D\phi_k) = 0$ *for some* $e_j \neq y_k \in \mathcal{F}(\phi_k)$*. Since* $\not<$ *is compact one can find a convergent subsequence* $y_{k_l} \to y \in \not<$*. Unless* $y = e_j$*, the uniform convergence of* $\phi_{k_l} \to \widehat{\phi} \in C^1(\Sigma \backslash \{e_1, e_2\}, \Sigma \backslash \{e_1, e_2\})$ *implies that* $\widehat{\phi}(y) = y$ *and* $\det(Id_{\mathbb{S}^n} - D\phi) = 0$ *at* y *- contradiction. However, since* t *is assumed to be bounded, the image of all* ϕ_{k_l} *would have to be bounded away from* e_j*, so the convergence of fixed points* $y_{k_l} \to e_j$ *is likewise impossible.*

It is therefore sufficient to show that for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $||f - P||_{C^1} < \delta$ *implies* $||\widehat{\phi} - \left(\frac{f_1 - f_2}{||f_1 - f_2||}, -\frac{t(f_1) - t(f_2)}{||f_1 - f_2||}\right)||_{C^1} < \varepsilon$*.*

In fact, this is going to be true, as long as both $\widehat{\phi}$ *and its derivative are bounded. But that is clearly the case since, by assumption 3 for any* $S \in \Sigma$*,* $P_1(S) \neq P_2(S)$*. Together with the assumption 4 this implies that there exists* $\eta > 0$ *such that for any* $S \in \Sigma$*,* $||P_1(S) - P_2(S)|| \geq \eta$ *and both* $||f_1 - f_2||$ *and* $|t(f_1) - t(f_2)|$ *are bounded from above by assumption. I shall take an open neighborhood* $V \subset K$ *of* P *defined by* $V = \{f \in K : ||f - P||_{C^1} < \delta\}$*. It is* *not hard to show that as $\delta \to 0$ $||f_1(S) - f_2(S)||$ has to likewise be bounded away from zero, which, together with the boundedness conditions of assumption 1 indeed ensures uniform convergence of* $\left(\frac{f_1-f_2}{||f_1-f_2||}, -\frac{t(f_1)-t(f_2)}{||f_1-f_2||}\right)$ *to* $\widehat{\phi}$ *in* C^1 *topology* ■

It has thus been shown that one is justified in concentrating, primarily, on regular economies. For these, the following general result is true.

Theorem 7 *Under assumptions 1-6 in regular two-jurisdiction economies the number of sorting equilibria is finite and even.*[4]

Proof. The proof of finiteness of the number of regular fixed points is standard (see Dierker 1974). The unusual part is the evenness, which follows almost immediately from the fact that Σ is homeomorphic to $\mathbb{S}^n$, which is a compact polyhedron with the Euler characteristic $\chi(\mathbb{S}^n) = \begin{cases} 2, & n \text{ - even} \\ 0, & n \text{ -odd} \end{cases}$. Furthermore, from the weak Pareto condition (assumption 5) I know that $\widehat{\phi}(S) \neq -S$. Consequently, for any $t \in [0,1]$ $||tS + (1-t)\widehat{\phi}(S)|| \neq 0$. Therefore, one can construct a homotopy map

$$H(S,t) = \frac{tS + (1-t)\widehat{\phi}(S)}{||tS + (1-t)\widehat{\phi}(S)||}$$

But this implies that the Lefschetz number of the mobility map $\Lambda\left(\widehat{\phi}\right) = \chi(\mathbb{S}^n)$ is even. But $\Lambda\left(\widehat{\phi}\right) = \sum_{\tau \in \mathcal{F}(\phi)} ind_\tau \widehat{\phi}$ which implies that for any regular $\widehat{\phi}$ the number of fixed points is indeed even (though possibly zero). As the e_j, by construction, are the only non-equilibrium fixed points, the conclusion follows ■

4 Discussion and an Example

In this paper it is shown that, subject to a generic regularity condition on policy functions, there is a finite and *even* number of sorting equilibria in the models of Caplin and Nalebuff (1997) and Gomberg (2004).

When jurisdictions are *ex ante* identical, the evenness of the number of sorting equilibria actually follows trivially from symmetry, since for each sorting equilibrium with the equilibrium policy profile (x_1^*, x_2^*) the profile (x_2^*, x_1^*) would also be an equilibrium. The real application of the result comes in models where the jurisdictions are intrinsically different in their policy rules P_j, such as would be the case if political parties use different internal decision-making rules to set their platforms, or when different towns face different technological or political

[4] To ensure actual sorting equilibrium existence additional assumptions may be needed (see Caplin and Nalebuff 1997).

constraints in making the decisions about local policies, such as taxes and local public good provision levels.[5] This paper shows that the evenness has nothing to do with the symmetry of the model.

Indeed, consider the standard one-dimensional spatial model of political competition, in which individuals have Euclidean preferences over policies so that the utility agent of type α obtains in a party with platform x is

$$u(\alpha, x_j) = -(\alpha - x)^2$$

Let the ideal points α be distributed on $[0,1]$ with density $f(\alpha) = 2\alpha$. and voters simply join the party the platform of which they like the most. Suppose that $X_1 = X_2 = (0,1)$, but that the parties choose their policies differently: party 1 chooses the ideal policy of its median voter, whereas party 2 chooses the ideal policy of its mean voter. Note that both policy rules fail to satisfy assumption 4*, but can be easily shown to satisfy assumptions 4 and 5. Assumptions 1-3 are likewise clearly satisfied. This example is asymmetric both in terms of *ex ante* difference between parties and the asymmetry of population distribution. It is not hard to check by direct computation that exactly two sorting equilibria will obtain in this case. In fact, in one equilibrium party 1 (choosing the median) locates to the right, choosing the policy $x_1^* = \sqrt{\frac{16}{23}} \approx 0.834$ and party 2 (choosing the mean) is relatively leftist, choosing the policy $x_2^* = \sqrt{\frac{4}{23}} \approx 0.417$, with the boundary between parties being at $\sqrt{\frac{9}{23}} \approx 0.625$. The second equilibrium has the first party to the left, with a policy $x_1^* = \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{46}\sqrt{\left(2001 + 552\sqrt{2}\right)}\right)\sqrt{2} \approx 0.457$ and the second party to the right, with policy $x_2^* = \frac{2}{3}\frac{1-\left(-\frac{1}{2}+\frac{1}{46}\sqrt{\left(2001+552\sqrt{2}\right)}\right)^3}{1-\left(-\frac{1}{2}+\frac{1}{46}\sqrt{\left(2001+552\sqrt{2}\right)}\right)^2} \approx 0.836$ and the boundary between parties at $-\frac{1}{2} + \frac{1}{46}\sqrt{\left(2001 + 552\sqrt{2}\right)} \approx 0.647$. It is straightforward to check, by solving for a general boundary between parties, that there are, indeed, no other sorting equilibria. Of course, the direct computation would not be possible for most population distributions, but the number of sorting equilibria is even in every regular model.

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