Pricing Contingent Claims Under Incomplete Information

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Abstract

We study the effects of incomplete information on the price of contingent claims within several frameworks, including when some additional sources of information other than securities histories can be used. The analysis is particularly relevant for emerging markets with young and sometimes weak monitoring institutions where the availability and accuracy of information are lower than in more mature markets. For a Black and Scholes (1973) economy and a two-factor set up a la Gibson and Schwartz (1990), we obtain closed form solutions for a futures contract and a European call option, when investors have normally distributed beliefs about some constant but non-observable parameter of the dividend rate or convenience yield. We characterize the shadow value of information and show that the volatility of the stock increases the derivative value but reduces learning. Moreover, the higher the volatility of the beliefs, the higher the value of the derivatives. Numerical simulations reveal that the impact of information accuracy can be significant, in particular when the option is out of the money. The case when non-observable stochastic fundamentals is also examined.

JEL classification: C11, D83, G13.

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1 Introduction

The seminal articles by Black and Scholes (1973) and Merton (1973) were the breakthrough the finance community had long been waiting for in order to be able to determine the fair price of contingent claims by no-arbitrage. Then, Harrison and Kreps (1979) inspired by the work of Cox and Ross (1976) introduce the notion of risk-neutral probability that allows to price contingent claims in a very general, possibly non-Markovian framework.

However, in real life, some characteristics of a stock are hardly known with perfect accuracy. This may be particularly relevant for emerging markets with weak monitoring institutions and for which reports on earnings by companies and records of information are not as comprehensive as in many mature markets.

The objective of this article is to shed some additional light on the effects of incomplete information and thus learning on the pricing of contingent claims by no-arbitrage. In particular we study the effects of additional sources of information outside market securities histories and the impact of the precision of the information on the price of derivatives. This work builds on prior research that has recognized the importance of learning about investment opportunities for an investor. We now review the existing literature on the topic and present the main findings of the paper.

1.1 Related Literature

Introducing incomplete information about some non-observable fundamentals of the economy, Detemple (1986), Dothan and Feldman (1986) and Gennote (1986) show that in a Markovian framework a separation principle holds: Agents first solve an inference problem to form their expectations, and second solve their dynamic optimization problem under the inferred information structure, incorporating learning as they update their beliefs. Within a general equilibrium framework, Veronesi (2000) investigates the effects of incomplete information on the equilibrium price of risky assets, equilibrium interest rate and its implications for the equity premium for CRRA utility investors. This latter can increase or decrease depending whether the coefficient of risk aversion is below or above unity. Some of the central issues of this paper are closely related to the work by David and Veronesi (2000). These authors develop a general equilibrium model of a pure exchange economy in which the average dividend growth rate is unknown but
is known to be either high or low with some switching Poisson probabilities. They apply their framework to price contingent claims, in particular focusing on the implied volatility. Babbs and Selby (1998) provide some technical conditions on normalized security prices under which it is possible to price contingent claims whose payoffs only depend on the sub-filtration generated by normalized asset prices even though markets are incomplete due to some additional sources of information other than market securities histories. In the sequel, we restrict our attention to this last class of contingent claims.

1.2 Results

We assume that agents cannot observe the convenience yield associated to some commodity in the classical Black-Scholes economy (model 1) or cannot observe the long run mean of the dividend rate or convenience yield when the latter follows a mean reverting (Ornstein-Uhlenbeck) process in a framework a la Gibson and Schwartz (1990) (model 2). We examine the impact of incomplete information on the price of several derivatives when investors have normally distributed beliefs. We characterize the shadow price of information and study the effect of uncertainty and the beliefs on the price of a futures contract and a European call option. For these two derivatives, we show that there is no direct gain from waiting. Incomplete information only affects the price of the contract through the correlation between the spot price, other observed signals and the beliefs. The result holds for any European type contract as long as the non-observable fundamental \( \mu \) is constant and received signals are Itô processes whose diffusion terms are independent from \( \mu \) and the drift terms are affine functions of \( \mu \). We find that the spot price volatility has two opposite effects. On the one hand, as in the classical Black-Scholes framework, the volatility enhances the value of the option. On the other hand, the volatility damages the quality of the signal and therefore reduces the learning effect and its associated value. Overall, the volatility effect always dominates the learning effect. The variance of the beliefs is a measure of the accuracy of the information. We show that the higher is the greater is value of the call and the futures. We extend our analysis to the case when the non-observable fundamental of the economy can be a stochastic process and show that investors’ Bayesian estimate do not necessarily learn the truth.

The paper is organized as follows. Section 2 describes the economic setting and provides some analytical results about the effects of incomplete information on the price of some contingent
claims when non-observable parameters are constant. Section 3 extends the analysis to the case where non-observable fundamentals are stochastic processes. Section 4 concludes. Proofs of all results are collected in the appendix.

2 The Economic Setting

2.1 Information Structure and Filtering Problem

Uncertainty is modeled by a probability space \((\Omega, \mathcal{F}, P^w)\) on which is defined a two dimensional (standard) Brownian motion \(w = (w_1, w_2)\). A state of nature \(\omega\) is an element of \(\Omega\). \(\mathcal{F}\) denotes the tribe of subsets of \(\Omega\) that are events over which the probability measure \(P^w\) is assigned. The standard Brownian motions \(w_1\) and \(w_2\) are possibly correlated and we denote by \(\rho\) the coefficient of correlation. Under the risk neutral probability measure \(P^w\), the value of the stock \(S\) is given by

\[
dS(t) = S(t) ((r - \delta)dt + \sigma_1 dw_1(t)),
\]

where \(dw_1(t)\) is the increment of the standard Wiener process under \(P^w\). There is some incomplete information about \(\delta\) and in the sequel we consider two types of frameworks.

2.1.1 Model 1

In model 1, \(\delta\) represents the convenience yield of some commodity. As exposed in Brennan and Schwartz (1985) “the convenience yield is the flow of services that accrues to an owner of the physical commodity but not to the owner of a contract for future delivery of the commodity”. We assume that \(\delta\) is known to be constant but cannot be observed. Investors observe the spot price itself and an additional signal \(A\) whose law of motion is given by

\[
dA(t) = A(t) (\lambda dt + \sigma_2 dw_2(t)),
\]

where \(dw_2(t)\) is the increment of the standard Wiener process under \(P^w\) and \(\lambda\) and \(\sigma_2\) are known parameters. The realizations of \(w_2\) can be observed but not those of \(w_1\). Let \(\mathcal{F}_t\) be the \(\sigma\)-algebra generated by the observations of the value of the spot price and signal \(A\), i.e., \(\mathcal{F}_t = \{S(s), A(s); 0 \leq s \leq t\}\) and augmented. At time \(t\), investors’ information set is \(\mathcal{F}_t\). The filtration \(\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{R}_+\}\) is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented). At time \(t\), we assume that investors’ believe that \(\mu\) is normally
distributed with mean $\Delta(t) = E^P [\delta | \mathcal{F}_t]$ and variance $\gamma(t) = E^P [(\delta - \Delta(t))^2 | \mathcal{F}_t]$, where $P$ denotes the probability measure representing the beliefs of the investors.

### 2.1.2 Model 2

Model 2 uses a framework a la Gibson and Schwartz (1990) and Schwartz (1997, models 2 and 3) and we assume that agents can observe both the spot price $S$ and $\delta$ that either represents the dividend rate of some financial asset or the convenience yield of some commodity. It is public knowledge that it follows a mean reverting process

$$d\delta(t) = \kappa(\alpha - \delta(t))dt + \sigma dw_1(t),$$

where the parameter $\kappa > 0$ measures the speed at which the process $\delta$ converges to its long run mean $\alpha$. Agents know that $\alpha$ is a constant but ignore its true value. Note that in contrast with model 1, the realizations of $w_1$ can be observed but not those of $w_2$. Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by the observations of the value of the spot price and the convenience yield, $\{S(s), \delta(s); 0 \leq s \leq t\}$ and augmented. At time $t$, investors’ information set is $\mathcal{F}_t$. The filtration $\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented). At time $t$, we assume that investors believe that $\alpha$ is normally distributed with mean $\alpha(t) = E^P [\alpha | \mathcal{F}_t]$ and variance $\gamma(t) = E^P [(\alpha - \alpha(t))^2 | \mathcal{F}_t]$.

Using Bayes’ rules, the evolution across time of the posterior probability $P$ is given by the following lemma.

**Lemma 1** In model 1, the law of motion of the posterior beliefs $P$ is given by

$$d\Delta(s) = \frac{\gamma(s)}{\sigma_1(1 - \rho^2)} (-d\bar{w}_1(s) + \rho d\bar{w}_2(s)),$$

$$\gamma(s) = -\frac{\gamma^2(s)}{\sigma_1^2(1 - \rho^2)},$$

where

$$d\bar{w}_1(s) = \frac{1}{\sigma_1 S(s)} (dS(s) - E^P [dS(s) | \mathcal{F}_s])$$

$$= dw_1(s) + \frac{\Delta(s) - \delta}{\sigma_1} ds,$$

$$d\bar{w}_2(s) = dw_2(s),$$
are the increments of a two-dimensional standard Wiener process under $P$, relative to the filtration $\mathcal{F}$. In model 2, the law of motion of the posterior beliefs $P$ is given by

$$
\begin{align*}
\frac{d\alpha(s)}{ds} &= \frac{\kappa\gamma(s)}{\sigma_2(1-\rho^2)}(-\rho d\bar{w}_1(s) + d\bar{w}_2(s)) \\
\gamma'(s) &= -\frac{\kappa^2\gamma^2(s)}{\sigma_2^2(1-\rho^2)},
\end{align*}
$$

where

$$
\begin{align*}
\bar{d}w_1(s) &= dw_1(s) \\
\bar{d}w_2(s) &= dw_2(s) + \frac{\kappa}{\sigma_2}(\alpha - \alpha(s))ds,
\end{align*}
$$

are the increments of a two-dimensional standard Wiener process under $P$, relative to the filtration $\mathcal{F}$.


Under the investor information structure the mean of beliefs is a martingale. This is a general result due to the fact that the non-observable parameter is constant. In model 1, the variance $\sigma_1^2$ measures the quality of the signal received. We notice that the additional signal is all the more informative that it is highly correlated with the spot price. The variance $\gamma$ is a measure of the precision of the knowledge about $\delta$ or $\alpha$. Note that it is deterministic, decreasing over time as knowledge about the true value of the non-observable parameter improves. Changes in $\gamma$ are negatively related with $\sigma_1$ and positively related with $\rho$: When the quality of the signal is poor, little information can be extracted and therefore, it takes a long time to get an accurate estimation of $\delta$. Changes in mean $\Delta$ are increasing in $\gamma$ (when $\gamma$ is high, a lot remains to be learned so beliefs can be revised at a faster speed) and decreasing with $\sigma_1$. In model 2, the variance $\sigma_2^2$ has the same effects as $\sigma_1^2$ has on the evolution of the beliefs in model 1. Note that although $[dS(t), d\alpha(t)] = 0$ the correlation between the spot price $S$ and the dividend rate/convenience yield $\delta$ allows a higher learning speed encapsulated by the term $\frac{\kappa^2(1-\rho^2)}{\sigma_2^2(1-\rho^2)}$, which is increasing in the instantaneous correlation $\rho$ between the Brownian motions $w_1$ and $w_2$. No other characteristic of $S$ impacts the dynamics of the beliefs. In addition, the adjustment speed $\kappa$ of the mean reverting process $\delta$ accelerates the decline of the variance $\gamma$, thus allowing a higher learning speed.

\footnote{$[X,Y]$ denotes the quadratic covariation between the stochastic processes $X$ and $Y$.}
In model 1, if we assume that financial markets consist of a riskless bond with constant interest rate \( r \) and a unique risky asset \( S \), it follows that we have two sources of uncertainty \((w_1, w_2)\) but only one risky asset so markets are incomplete. Hence we may not be able to price contingent claims. However, Babbs and Selby (1998) show that the class of contingent claims whose payoffs under the risk neutral measure only depend on the sub-filtration generated by normalized securities prices can priced in the “usual” way provided that the stochastic differential equation describing the law of motion of securities under the risk neutral measure admits a unique (weak) solution. In this paper, such conditions are satisfied. Moreover, in order to rule out arbitrage opportunity, we make the following assumption.

**Assumption A1.** Investors have common beliefs about the true value of the non-observable parameter.

We develop two distinct approaches to compute the price of derivative contracts. The first method relies on a change of probability measure under which the value of the contract is re-expressed and easier to compute. The second one is based on the law of iterated expectations. Since we restrict our attention to European contracts (the result will not hold in general for American contracts), if \( P(t; \mu_t) \) denotes the price at time \( t \) of a derivative contract with underlying security \( S \) when the non-observable fundamental (possibly a stochastic process) \( \mu_t \) is known, then by the law of interacted expectations, at time \( t \), the price \( P(t) \) of the derivative under incomplete information is given by

\[
P(t) = E \left[ P(t; \mu_t) \right],
\]

where the last expectation is taken over the random variable \( \mu_t \) with \( \mu_t \sim N(m_t, \gamma_t) \).

We start with a general result regarding the value of information. We impose the following regularity conditions on the price \( P(\mu) \) of a European contract under perfect information.

**Assumption A2.** For all values of \( \mu \), \( P(\mu) \) and its derivatives \( P^k(\mu) \) for \( k = 1, 2 \) are smooth functions that satisfy some growth conditions

\[
\lim_{|x| \to \infty} P^k(x) e^{-\frac{ax^2}{2}} = 0 \text{ for all } a > 0.
\]

**Proposition 1** Under assumption A2., when agents cannot observe the true value of \( \mu \) and at time \( t \) have normally distributed beliefs \( N(m_t, \gamma_t) \) about some non-observable fundamental of the
economy \( \mu \), then the price of the contract \( Q \) satisfies
\[
\frac{1}{2} Q_{mm} = Q_\gamma.
\]

**Proof.** See appendix 1. ■

In the next section, we determine the price of a European call option and investigate the impact of incomplete information.

### 2.2 European Call Option

We start this section by explaining in details how the first method works choosing for exposure reasons the framework of model 1. Let \( P_\delta \) be the probability measure under which \( \delta \) is known and define the processes \( \eta_\delta \) and \( \xi_\delta \) by
\[
\eta_\delta(t) = \frac{\Delta(t) - \delta}{\sigma_1(1 - \rho^2)} (1, -\rho),
\]
and
\[
\xi_\delta(t) = \exp \left(-\int_0^t \gamma_\delta(s)^\top dw(s) - \frac{1}{2} \int_0^t \|\gamma_\delta(s)\|^2 ds\right).
\]
\( \xi_\delta \) is the density process of the Radon-Nikodym derivative\(^2\) of \( P \) with respect to \( P_\delta \), i.e.,
\[
\xi_\delta(t) = \frac{dP(t)}{dP_\delta(t)}.
\]

The following lemma provides an expression of \( \xi_\delta \) as a function of the mean \( \Delta \) and the variance \( \gamma \).

**Lemma 2** The Radon-Nikodym derivative \( \xi_\delta \) is given by
\[
\xi_\delta(\Delta(t), \gamma(t)) = \sqrt{\frac{\gamma(t)}{\gamma_0}} \exp \left( \frac{1}{2\gamma(t)} (\Delta(t) - \delta)^2 - \frac{1}{2\gamma_0} (\Delta_0 - \delta)^2 \right).
\]

**Proof.** See appendix 2. ■

Note the density process of the Radon-Nikodym is nothing but the inverse of the density of a normally distributed random variable with mean \( \Delta(t) \) and variance \( \gamma(t) \), normalized such \( \xi_\delta(0) = 1 \). In the sequel, \( E^\delta [ \cdot | \mathcal{F}_t ] \) denote the conditional expectation under the probability measure \( P_\delta \). We now compute the price of a European call option on \( S \) with strike price \( K \) and maturity date \( T \).

\(^2\)In our framework, for \( X \) in \( \mathbb{R}^2 \), \( \|X\|^2 = X^\top \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} X \).
2.3 A Black-Scholes’ formula under incomplete information (Model 1)

The price $C$ of a European call option on $S$ with strike price $K$ and maturity date $T$ satisfies the following PDE

$$rC = C_t + (r - \Delta)SC_S + \frac{\sigma^2}{2}S^2C_{SS} - S\gamma C_{S\Delta} + \frac{\gamma^2}{\sigma^2(1-\rho^2)} \left( \frac{1}{2}C_{\Delta\Delta} - C_\gamma \right),$$  \hspace{1cm} (1)

subject to the terminal boundary condition

$$C(S, \Delta, \gamma, T) = [S - K]_+, \hspace{1cm} \text{where } x_+ = \max(x, 0).$$

**Interpretation**  As usual, the return of investing an amount $C$ into a safe asset with constant rate of return $r$ must be equal to the expected capital gain from waiting (since no dividend is paid) governed by the changes in the value of the stock, time and the beliefs. The three first terms on the RHS of relationship (1) are the usual ones (given a fixed value for $\Delta$) and represent the expected change in the option value as $S$ and time vary. Appearing in the two last terms, $\frac{\gamma^2}{\sigma^2(1-\rho^2)} \left( \frac{1}{2}C_{\Delta\Delta} - C_\gamma \right)$ represents the shadow price of information. The direct gain from learning is equal to zero as the amount of information conveyed by the mean process $\frac{\gamma^2}{\sigma^2(1-\rho^2)}$ exactly cancels out with the reduction in variance $-\frac{\gamma^2}{\sigma^2(1-\rho^2)}$. The intuition behind this result is that in contrast with an American option, there is no action (early exercise) the holder of the contract can take (besides selling the option) before the maturity date. Since all investors are assumed to have the same information and therefore update their beliefs in the same fashion, selling the call before the expiration date does provide any gain. By Theorem 12.7. p 36 in Liptser and Shiryaev (2000, volume II) that assumes that the non-observable parameter is a random variable $\theta$ and the signal received is an Ito process whose diffusion term is independent from $\theta$ and drift term is an affine function of $\theta$ and proposition 1., we can conclude that the direct gain from learning is zero for all European contracts. The median term $-S\gamma C_{S\Delta}$ in relationship (1) represents the effects of change of the mean $\Delta$ on the marginal value of the option due to the correlation between the value of the stock and the mean. The sign of the cross derivative $C_{S\Delta}$ is somewhat difficult to predict.

The benefits of learning depends on the remaining time until the expiration date $T$. Nevertheless, one can note that in the case where $\delta$ is known, the hedge ratio $\frac{\partial C}{\partial S}$ is decreasing in $\delta$. When $\Delta$ increases, this somehow corresponds to a decrease in the perceived value of $\delta$. 


This intuitive reasoning leads us to conjecture that $C_{S\Delta}$ must be negative. This median term represents the indirect effect of learning.

It is apparent that the magnitude of the uncertainty $\sigma_1$ now plays an ambiguous role. On the one hand, an increase in $\sigma_1$ rises the option value as in the complete information case. On the other hand, when $\sigma_1$ increases, less information can be extracted from the observations of $S$ and therefore, it lowers the option value by decreasing the amount of information contained in the signal.

The price $C$ of a European call option on $S$ with strike price $K$ and maturity date $T$ is given by

$$C(S_t, \Delta_t, \gamma_t, t) = E^{\mathcal{P}} \left[ (S(T) - K)^+ e^{-r(T-t)} \mid \mathcal{F}_t \right].$$

Using the Radon Nikodym derivative theorem, the price $C$ can be rewritten

$$C(S_t, \Delta_t, \gamma_t, t) = E^{\mathcal{P}} \left[ \xi_{\delta}(T) \left( S(T) - K \right)^+ e^{-r(T-t)} \mid \mathcal{F}_t \right]$$

$$= \sqrt{\frac{\gamma(T)}{\gamma_t}} e^{-\frac{1}{2\sigma_1^2}(\Delta(t)-\delta)^2} E^{\mathcal{P}^\delta} \left[ e^{\frac{1}{2\sigma_1^2}(\Delta(T)-\delta)^2} \left( S(T) - K \right)^+ e^{-r(T-t)} \mid \mathcal{F}_t \right].$$

Note that under the probability measure $P_\delta$, the laws of motion of $S$ and $\Delta$ are given by

$$dS(t) = S(t) \left[ (r - \delta)dt + \sigma_1 dw(t) \right]$$

$$d\Delta(t) = \frac{\gamma(t)}{\sigma_1(1 - \rho^2)} \left( -\frac{1}{\sigma_1} (\Delta(t) - \delta) dt - dw_1(t) + \rho dw_2(t) \right).$$

**Proposition 2** When incomplete information is introduced on the dividend yields $\delta$ and investors have common normally distributed beliefs $(\Delta_t, \gamma_t)$, the price $C$ of a European Call option at time $t$ with strike price $K$ and maturity date $T$ is given by

$$C(S_t, \Delta_t, \gamma_t, t) = e^{-\frac{\Delta(t)}{2} + \frac{\gamma(t)}{2}(T-t)^2} S_t N(d_1') - e^{-r(T-t)} K N(d_2'),$$

where

$$d_1' = \frac{\ln\left( \frac{S_t}{K} \right) + \left( r - \Delta_t + \gamma_t(T-t) + \frac{\sigma_1^2}{2} \right)(T-t)}{\sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}}$$

$$d_2' = \frac{\ln\left( \frac{S_t}{K} \right) + \left( r - \Delta_t - \frac{\sigma_1^2}{2} \right)(T-t)}{\sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}}$$

and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$. 

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Proof. See appendix 3. ■

The additional signal $A$ does not have any direct impact on the price on the call option. It only affects the learning speed through the dynamics of the mean $\Delta$ and the variance $\gamma$ of beliefs.

2.4 Computing the Greeks

In this section, we compute the traditional dependence of the call option with respect to the spot price (Delta and Gamma), the asset volatility (Vega) decomposing the effect in two parts, the volatility effect and the learning effect, the time to maturity $\tau = T - t$ (Theta) and interest rate (Rho). Then, we investigate the impact of the volatility (interpreted as a measure of the accuracy of the information gathered) on the price of the call. The following table summarizes the value of the Greeks.

\[
\begin{align*}
\text{Delta} & = \frac{\partial C}{\partial S} = e^{-\Delta(t-T)+\frac{\gamma}{2}(T-t)^2} N(d_1') > 0 \\
\Gamma & = \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\Delta(t-T)+\frac{\gamma}{2}(T-t)^2} n(d_1')}{S_t \sqrt{\sigma_1^2 + \gamma (T-t)^2}} > 0 \\
\Theta & = \frac{\partial C}{\partial \tau} = (-\Delta + \tau \gamma) e^{-\delta T + \frac{\gamma}{2} T^2} S_t N(d_1') + re^{-r\tau} K N(d_2') + \\
& \quad S_t e^{-\Delta + \frac{\gamma}{2} \tau} n(d_1') \frac{\sigma_1^2 + 2 \gamma \tau}{2 \sqrt{\sigma_1^2 + \gamma T^2 \tau}} > 0 \\
\nu & = \frac{\partial C}{\partial \sigma_1^2} = S_t e^{-\Delta(T-t)+\frac{\gamma}{2}(T-t)^2} \frac{(T-t)}{2 \sqrt{\sigma_1^2 + \gamma (T-t)^2}} n(d_1') > 0 \\
Rho & = \frac{\partial C}{\partial r} = (T-t) e^{-r(T-t)} K N(d_2') > 0.
\end{align*}
\]

Proof. See appendix 4. ■

We notice that under incomplete information the sign of the Greeks remains the same as under complete information, besides maybe Theta. In this later case, the learning effect has a positive contribution as far as the impact of passage of time is concerned. For Vega, the role of the volatility $\sigma_1^2$ is now twofold. On the one hand, as in the classical Black-Scholes framework, the volatility enhances the value of the option. On the other hand, the volatility $\sigma_1^2$ damages the quality of the signal and therefore reduces the learning effect and its associated value. More precisely, as derived in appendix 5, the marginal decrease into the learning component of the
price of the call option is given by
\[
\frac{\partial C}{\partial \sigma_t^2} = -S_t e^{-\Delta_t(T-t) + \gamma_t(T-t)^2} \left( n(d'_1) \frac{\gamma_t(T-t)}{2\sigma_t^2 \sqrt{T-t} \sqrt{1 + \gamma_t(T-t)}} \right) < 0.
\]

The fact that Vega is positive indicates that the volatility effect always dominates the learning effect. We now explore the impact of the precision of the beliefs on the price of the call. We have the following proposition.

**Proposition 3** The impact of the volatility of beliefs on the price of the call is given by
\[
\frac{\partial C}{\partial \gamma_t} = S_t e^{-\Delta_t(T-t) + \frac{\gamma_t(T-t)^2}{2}} \left( n(d'_1) + \frac{n(d'_1)}{\sigma_t \sqrt{T-t} \sqrt{1 + \gamma_t(T-t)}} \right) > 0.
\]

The higher the variance \( \gamma \) (lack of precision about the true value of the dividend yield), the higher the price of the call \( C \).

**Proof.** See appendix 5.  

When there is a lot imprecision about the true value of the dividend yield, the investor has some hope that it can be small and the odds are greater than \( S \) ends up above \( K \) at time \( T \). Given the convexity of the payoff function and the fact that only the truncated (at the strike price \( K \)) distribution of the stock price matters at time \( T \), the value of the call option \( C \) increases when the variance of beliefs \( \gamma \) goes up. Therefore, more information (precision) is not necessary a good thing. The value of the option can be decomposed into two components: One term equals to the value obtained using the Black-Scholes formula maintaining \( \Delta_t \) fixed, excluding learning plus one term purely due to learning, i.e.
\[
C(S_t, \Delta_t, \gamma_t, t) = e^{-\delta_t(T-t)}S_t N(d_1) - e^{-r(T-t)}K N(d_2)
\]
\[
+ e^{-\delta_t(T-t)}S_t \left( e^{\frac{1}{2}(T-t)^2} N(d'_1) - N(d_1) \right) - e^{-r(T-t)}K \left( N(d'_2) - N(d_2) \right)
\]
where
\[
d_1 = d_2 + \sigma_t \sqrt{T-t}
\]
\[
d_2 = \frac{\ln(S_t) + (r - \sigma_t^2/2)(T-t)}{\sigma_t \sqrt{T-t}}.
\]

Note that since \( C_\gamma > 0 \), the learning term is always positive. We conclude this section by presenting some numerical results in order to quantify the impact of incomplete information.
2.5 Numerical Simulations

Table I: Impact of the precision of beliefs on the price of a European call option

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$S/K$</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
<th>1.7</th>
<th>1.9</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$C =$</td>
<td>0.01</td>
<td>0.04</td>
<td>0.14</td>
<td>0.36</td>
<td>0.73</td>
<td>1.23</td>
<td>1.87</td>
<td>3.41</td>
<td>5.16</td>
<td>6.99</td>
<td>8.86</td>
</tr>
<tr>
<td>0.05</td>
<td>$C =$</td>
<td>0.27</td>
<td>0.56</td>
<td>0.96</td>
<td>1.47</td>
<td>2.09</td>
<td>2.78</td>
<td>3.54</td>
<td>5.22</td>
<td>7.05</td>
<td>8.97</td>
<td>10.94</td>
</tr>
<tr>
<td>0.1</td>
<td>$C =$</td>
<td>0.75</td>
<td>1.24</td>
<td>1.85</td>
<td>2.55</td>
<td>3.33</td>
<td>4.17</td>
<td>5.06</td>
<td>6.97</td>
<td>9.00</td>
<td>11.10</td>
<td>13.26</td>
</tr>
<tr>
<td>0.15</td>
<td>$C =$</td>
<td>1.33</td>
<td>2.01</td>
<td>2.80</td>
<td>3.66</td>
<td>4.60</td>
<td>5.59</td>
<td>6.62</td>
<td>8.78</td>
<td>11.05</td>
<td>13.38</td>
<td>15.76</td>
</tr>
<tr>
<td>0.2</td>
<td>$C =$</td>
<td>1.99</td>
<td>2.85</td>
<td>3.81</td>
<td>4.85</td>
<td>5.94</td>
<td>7.08</td>
<td>8.26</td>
<td>10.70</td>
<td>13.24</td>
<td>15.84</td>
<td>18.48</td>
</tr>
</tbody>
</table>

$S = 10, r = 0.05, \sigma_1 = 0.2, \delta = \Delta = 0.03, T = 2$

Table I presents the effects of the variance of beliefs on the price of a European call option when the mean of the beliefs and the true value coincide. Recall that when $\gamma = 0$, we are in the case of complete information and therefore the price of the European call option is given by the standard Black-Scholes formula. Worth noticing is the large impact of the variance when the option is out of the money. This is due to the convexity of the payoff of the call option.

2.6 A Black-Scholes’ formula under incomplete information (Model 2)

In this section, we use the second method to express the price of a European call option. The following proposition recalls the value of a European call in a framework a la Gibson and Schwartz (1990) under perfect information.

**Proposition 4** Under perfect information, the price $C$ of a European call option on $S$ with strike price $K$ and maturity date $T$ satisfies the following PDE

$$rC = C_t + (r - \delta)SC_S + \frac{\sigma_1^2}{2}S^2C_{SS} + \kappa(\alpha - \delta)C_\delta + \frac{\sigma_2^2}{2}C_{\delta\delta} + \rho\sigma_1\sigma_2SC_S$$

with terminal boundary condition $C(S, \delta, T) = [S - K]^+$. The solution of this equation is given by

$$C(S_t, \delta_t, t) = e^{m(\delta_t, T - t; \alpha) + \frac{\Sigma(T-t)}{2}}S_tN(d_1) - e^{-r(T-t)}KN(d_2),$$
where

\[ m(\delta_t, T - t; \alpha) = \frac{\sigma_1^2(T - t) - 1 - e^{-\kappa(T - t)}}{\kappa} \delta_t - \left( T - t - \frac{1 - e^{-\kappa(T - t)}}{\kappa} \right) \alpha. \]

\[ \Sigma(T - t) = \sigma_1^2(T - t) + \sigma_2^2 \int_0^{T-t} \left( 1 - e^{-\kappa(T-t-s)} \right)^2 \, ds - \frac{2\rho \sigma_1 \sigma_2}{\kappa} \left( T - t - \frac{1 - e^{-\kappa(T - t)}}{\kappa} \right). \]

and

\[ d_1 = d_2 + \sqrt{\Sigma(T - t)} \]
\[ d_2 = \frac{\ln(S_t/K) + m(\delta_t, T - t; \alpha)}{\sqrt{\Sigma(T - t)}}. \]

**Proof.** See appendix 6. \( \blacksquare \)

**Proposition 5** When investors have incomplete information about the long run value of the dividend rate/convenience yield the price \( C \) of a European call option on \( S \) with strike price \( K \) and maturity date \( T \) satisfies the following PDE

\[ rC = C_t + (r - \delta)SC_S + \frac{\sigma_1^2}{2} S^2 C_{SS} + \kappa(\alpha - \delta)C_\delta + \frac{\sigma_2^2}{2} C_{\delta\delta} + \rho \sigma_1 \sigma_2 S C_{S\delta} \]
\[ + \kappa \gamma C_\delta \alpha + \frac{\kappa^2 \gamma^2}{\sigma_2^2(1 - \rho^2)} \left( \frac{1}{2} C_{\alpha\alpha} - C_{\gamma} \right), \]

and with terminal condition \( C(S, \delta, \alpha, \gamma, T) = [S - K]^+. \) The solution of this equation is given by

\[ C(S_t, \delta_t, \alpha_t, \gamma_t, t) = e^{m(\delta_t, T-t; \alpha_t) + \frac{\gamma_t}{T-t} \left( T - t - \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2} S_t N(d_1') - e^{-r(T-t)} K N(d_2')}, \]

where

\[ d_1' = d_2' + \sqrt{\Sigma(T - t)} \sqrt{\Omega(T - t)} \]
\[ d_2' = \frac{\ln(S_t/K) + m(\delta_t, T - t; \alpha)}{\sqrt{\Sigma(T - t)} \sqrt{\Omega(T - t)}}, \]

and

\[ \Omega(T - t) = 1 + \frac{\gamma_t}{\Sigma(T - t)} \left( T - t - \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)^2. \]

**Proof.** The result can be obtained by easy but tedious substitution. \( \blacksquare \)
2.6.1 Comparative Statics

All the proofs are reported in appendix 6. The effects of incomplete information are similar to those described in model 1. The shadow value of information is equal to zero and a higher variance of the beliefs \( \gamma_t \) enhances the value of the call. As in the case of complete information, the effect of the variance \( \sigma_t^2 \) of the spot price can have an ambiguous effect. On the one hand, it rises the value of the call. On the other hand, due to the correlation between \( S \) and \( \delta \), an increase in \( \sigma_t^2 \) reduces the variance \( \Sigma(t, T) \) which shrinks the value of the call. More precisely, when \( T - t - \frac{\rho \sigma_t^2}{\kappa \sigma_1} (T - t - \frac{1-e^{-\kappa(T-t)}}{\kappa}) \) is positive, which is always the case when \( \rho \) is small, an increase in \( \sigma_1 \) rises the value of the call option. The effect is similar with the variance \( \sigma_2^2 \), the condition being replaced by \( \int_t^T \left( \frac{1-e^{-\kappa(T-s)}}{\kappa} \right)^2 ds - \frac{\rho \sigma_t^2}{\kappa \sigma_2} (T - t - \frac{1-e^{-\kappa(T-t)}}{\kappa}) \) positive, which again is always the case is \( \rho \) is small enough. As for model 1, we conclude this section by presenting some numerical results in order to quantify the impact of incomplete information.

2.7 Numerical Simulations

Table II: Impact of the precision of beliefs on the price of a European call option

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( S/K )</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
<th>1.7</th>
<th>1.9</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( C = )</td>
<td>0.007</td>
<td>0.031</td>
<td>0.095</td>
<td>0.221</td>
<td>0.426</td>
<td>0.72</td>
<td>1.10</td>
<td>2.10</td>
<td>3.35</td>
<td>4.75</td>
<td>6.24</td>
<td>7.00</td>
</tr>
<tr>
<td>0.05</td>
<td>( C = )</td>
<td>0.007</td>
<td>0.033</td>
<td>0.098</td>
<td>0.226</td>
<td>0.434</td>
<td>0.73</td>
<td>1.12</td>
<td>2.12</td>
<td>3.37</td>
<td>4.76</td>
<td>6.26</td>
<td>7.02</td>
</tr>
<tr>
<td>0.1</td>
<td>( C = )</td>
<td>0.008</td>
<td>0.034</td>
<td>0.102</td>
<td>0.232</td>
<td>0.442</td>
<td>0.74</td>
<td>1.13</td>
<td>2.14</td>
<td>3.38</td>
<td>4.78</td>
<td>6.27</td>
<td>7.04</td>
</tr>
<tr>
<td>0.15</td>
<td>( C = )</td>
<td>0.008</td>
<td>0.036</td>
<td>0.105</td>
<td>0.238</td>
<td>0.451</td>
<td>0.75</td>
<td>1.14</td>
<td>2.15</td>
<td>3.40</td>
<td>4.80</td>
<td>6.29</td>
<td>7.06</td>
</tr>
<tr>
<td>0.2</td>
<td>( C = )</td>
<td>0.009</td>
<td>0.037</td>
<td>0.108</td>
<td>0.243</td>
<td>0.459</td>
<td>0.76</td>
<td>1.15</td>
<td>2.17</td>
<td>3.41</td>
<td>4.81</td>
<td>6.31</td>
<td>7.07</td>
</tr>
</tbody>
</table>

\( S = 10, r = 0.05, \sigma_1 = 0.2, \sigma_2 = 0.2, \delta = \alpha = 0.03, \kappa = 0.1, \rho = 0.5, T = 2 \)

Table II presents the effects of the variance of beliefs on the price of a European call option when the mean of the beliefs and the true value coincide. As displayed in table II, the impact of incomplete information is small and most significant when the option is out of the money. Again, this is due to the convexity of the payoff of the call option. We now briefly examine the futures contracts for the two models.
2.8 Futures Contract

We have the following proposition.

**Proposition 6** In model 1, the price $F$ of a futures contract on commodity $S$ satisfies the following partial differential equation (PDE)

$$0 = F_t + (r - \Delta)SF_S + \frac{\sigma^2}{2} S^2 F_{SS} - S\gamma F_{S\Delta} + \frac{\gamma^2}{\sigma^2_1(1 - \rho^2)} \left( \frac{1}{2} F_{\Delta\Delta} - F_\gamma \right),$$

with terminal boundary condition $F(S, \Delta, \gamma, T) = S$. The solution to this equation is given by

$$F(S_t, \Delta_t, \gamma_t, t) = E^P \left[ S(T) \mid \mathcal{F}_t \right] = S_t e^{(r-\Delta)(T-t) + \frac{1}{2} \gamma^2(T-t)^2}.$$

In model 2, the price $F$ of a futures contract on asset $S$ satisfies the following PDE

$$0 = F_t + (r - \delta)SF_S + \frac{\sigma^2}{2} S^2 F_{SS} + \kappa(\alpha - \delta)F_\delta + \frac{\sigma^2}{2} F_{\delta\delta} + \rho\sigma_1\sigma_2SF_S$$

$$+ \kappa\gamma F_{\delta\alpha} + \frac{\kappa^2\gamma^2}{\sigma^2_2(1 - \rho^2)} \left( \frac{1}{2} F_{\alpha\alpha} - F_\gamma \right).$$

and with terminal boundary condition $F(S, \delta, \alpha, \gamma, T) = S$. The solution to this equation is given by

$$F(S_t, \delta_t, \alpha_t, \gamma_t, t) = E^P \left[ S(T) \mid \mathcal{F}_t \right] = S_t \exp \left( -\delta_t \frac{1 - e^{-\kappa(T-t)}}{\kappa} + A(\alpha_t, \gamma_t, T - t) \right),$$

where

$$A(\alpha_t, \gamma_t, T - t) = \left( r + \frac{\sigma^2}{2\kappa^2} - \frac{\rho\sigma_1\sigma_2}{\kappa} \right) (T - t) + \frac{\sigma^2}{4} \frac{1 - e^{-2\kappa(T-t)}}{\kappa^3}$$

$$+ \left( \rho\sigma_1\sigma_2 - \frac{\sigma^2_2}{\kappa} \right) \frac{1 - e^{-\kappa(T-t)}}{\kappa^2} - \alpha_t \left( T - t - \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)$$

$$+ \frac{\gamma t}{2} \left( T - t - \frac{1 - e^{-\kappa(T-t)}}{\kappa^2} \right)^2.$$

**Proof.** Using the second method readily gives the desired result. Alternatively, one can directly check the result by substitution\(^3\). ■

\(^3\)When the parameter $\alpha$ is known, the value of a futures contract is given in Schwartz (1997).
From the above expressions, it is easy to check that the shadow value of information $\frac{1}{2}F_{mm} - F_\gamma$ where $m \in \{\alpha, \Delta\}$ is equal to zero. In addition, we notice that the higher the variance of beliefs $\gamma_t$, the larger the value of the futures. For model 1, a higher estimation of $\delta$ reduces the value of the futures contract; in a similar fashion, in model 2, since $(T - t) - \frac{1 - e^{-\kappa(T-t)}}{\kappa} = \int_t^T (1 - e^{-\kappa(T-u)}) \, du > 0$, a higher estimation of $\alpha$ reduces the value of the futures contract. We now extend our analysis to the case where non-observable fundamentals are stochastic processes.

3 Case when non-observable fundamentals are stochastic processes

When the non-observable fundamentals are stochastic processes, in general the mean of beliefs no longer follow a martingale under the investor information structure. In this section, again, we consider the Gibson and Schwartz (1990) model for a commodity whose convenience yield is known to follow a mean reverting process but cannot be observed. This time, the investor information structure is different from the information structure that prevails under complete information. In the latter case, at time $t$, the information set is the $\sigma$-algebra generated by the observations of the value of the spot price and the convenience yield, i.e. $\mathcal{F}_t = \{S(s), \delta(s); 0 \leq s \leq t\}$ and augmented whereas in our case, the investor’s information set is the $\sigma$-algebra generated by the observations of the realizations of the spot price only, i.e. $\mathcal{F}^I_t = \{S(s); 0 \leq s \leq t\}$ and augmented. The filtration is $\mathbb{F}^I = \{\mathcal{F}_t^I, t \in \mathbb{R}_+\}$. At time $t$, investors are assumed to have normally distributed beliefs about $\delta(t)$. As shown in Liptser and Shiryaev (2000, volume I), denoting $\Delta(t) = E^P [\delta(t) \mid \mathcal{F}_t^I]$ and $\gamma(t) = E^P [\left(\delta(t) - \Delta(t)\right)^2 \mid \mathcal{F}_t^I]$, the solution to the inference problem is

\[
\begin{align*}
    dS(s) &= S(s) ((r - \Delta(s))ds + \sigma_1 d\bar{w}_1(s)) \\
    d\Delta(s) &= \kappa(\alpha - \Delta(s))ds + \left(\rho \sigma_2 - \frac{\gamma(s)}{\sigma_1}\right) d\bar{w}_1(s) \\
    \dot{\gamma}(s) &= -\frac{1}{\sigma_1^2} (\gamma(s) - \bar{\gamma}) \left(\gamma(s) - \gamma\right),
\end{align*}
\]

where

\[
\begin{align*}
    d\bar{w}_1(s) &= \frac{1}{\sigma_1 S(s)} \left(dS(s) - E^P [dS(s) \mid \mathcal{F}_s^I]\right) \\
    &= dw_1(s) + \frac{\kappa}{\sigma_1} (\Delta(s) - \delta(s)) \, ds,
\end{align*}
\]
and

\[ \bar{\gamma} = -(\kappa\sigma_1^2 - \rho\sigma_1\sigma_2) + \sigma_1 \sqrt{\sigma_1^2 \kappa^2 - 2\rho\kappa\sigma_1\sigma_2 + \sigma_2^2} \geq 0 \]

\[ \gamma = -(\kappa\sigma_1^2 - \rho\sigma_1\sigma_2) - \sigma_1 \sqrt{\sigma_1^2 \kappa^2 - 2\rho\kappa\sigma_1\sigma_2 + \sigma_2^2} \leq 0. \]

There are two cases where \( \gamma = 0 \): (i) \( \rho = -1 \) and (ii) \( \rho = 1 \) and \( \sigma_2 \geq \kappa\sigma_1 \). We notice that a higher reverting speed \( \kappa \) improves the learning speed. The direct effect of a high variance \( \sigma_1^2 \) is to lower the learning speed. The indirect effect due to the correlation between \( S \) and \( \delta \) depends on the sign of \( \rho \). When \( \rho \) is positive (negative) the indirect effect is to delay (fasten) learning. The indirect effect can be significant and prevent convergence of the estimate to the true process as shown in the sequel. The dynamics of variance \( \gamma \) are deterministic and given by a Riccati equation whose solution is

\[ \gamma(s) = \frac{\gamma - \gamma(t) - \frac{\gamma - \gamma(t)}{\sigma_1^2} e^{-\frac{\gamma - \gamma(t)}{\sigma_1^2}(s-t)}}{1 - \gamma(t) - \frac{\gamma - \gamma(t)}{\sigma_1^2} e^{-\frac{\gamma - \gamma(t)}{\sigma_1^2}(s-t)}}. \]

Depending on the initial condition at time \( t \), if \( \gamma_t \) is below (above) \( \bar{\gamma} \), the variance is increasing (decreasing) as time passes and in both cases we have

\[ \lim_{s \to \infty} \gamma(s) = \bar{\gamma}. \]

Hence, estimations do not converge to the true process. Asymptotically the true convenience yield admits a normal distribution

\[ \delta_\infty \sim N(\alpha, \sigma_2^2/\kappa), \]

whereas the estimator \( \Delta(t) = E^P [\delta(t) \mid F_t^t] \) has a distinct limiting normal distribution

\[ \Delta_\infty \sim N(\alpha, \sigma_2^2/2\kappa + \bar{\gamma}). \]

It has the same mean but a higher variance than the true distribution. We now study the direct gain from waiting.

**Proposition 7** In a framework a la Gibson and Schwartz (1990), when investors cannot observe the convenience yield of a commodity \( S \), the price \( C \) of a European call option on \( S \) with strike price \( K \) and maturity date \( T \) satisfies the following PDE

\[ rC = C_t + (r - \Delta)SC_S + \frac{\sigma_2^2}{2}SC_{SS} + \kappa(\alpha - \Delta)C_\Delta + (\rho\sigma_1\sigma_2 - \gamma)SC_{S\Delta} \]

\[ + \frac{1}{2} \left( \rho\sigma_2 - \frac{\gamma}{\sigma_1} \right)^2 C_{\Delta\Delta} - \left( 2(\kappa - \frac{\rho\sigma_2}{\sigma_1})\gamma + \frac{\gamma^2}{\sigma_1^2} - (1 - \rho^2)\sigma_2^2 \right) C_\gamma, \]
and with terminal condition \( C(S, \Delta, \gamma, T) = [S - K]^+ \). The solution of this equation is given by

\[
C(S_t, \Delta_t, \gamma_t, t) = e^{m(\Delta_t, T-t; \alpha) + \frac{\Sigma(T-t)}{2}} S_t N(d'_1) - e^{-r(T-t)} KN(d'_2),
\]

where

\[
d'_1 = d'_2 + \sqrt{\Sigma(T-t)} \sqrt{V(T-t)}
\]

\[
d'_2 = \frac{\ln(S_t/K) + m(\Delta_t, T-t; \alpha)}{\sqrt{\Sigma(T-t)} \sqrt{V(T-t)}},
\]

and

\[
V(T-t) = 1 + \frac{\gamma_t}{\Sigma(T-t)} \left( \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)^2.
\]

**Proof.** The result can be obtained by easy but tedious substitution. \( \blacksquare \)

In this framework, novelty comes from the value of information. As shown in appendix 7, we still have \( \frac{1}{2} C_{\Delta \Delta} = C_\gamma > 0 \) and the informativeness is given by

\[
\left( \rho \sigma_2 - \frac{\gamma}{\sigma_1} \right)^2 - \left( 2(\kappa - \frac{\rho \sigma_2}{\sigma_1}) \gamma + \frac{\gamma^2}{\sigma_1^2} - (1 - \rho^2) \sigma_2^2 \right) = \frac{\sigma_2^2}{2} - \kappa \gamma.
\]

It is positive exactly when

\[
\frac{\sigma_2^2}{2} \geq \kappa \gamma. \tag{2}
\]

Since \(-2\kappa \gamma + \sigma_2^2 = \left( \rho \sigma_2 - \frac{\gamma}{\sigma_1} \right)^2 \geq 0\), it follows that if the initial condition \( \gamma_t \) is below \( \overline{\gamma} \), the variance \( \gamma \) is increasing over time up to \( \overline{\gamma} \) and therefore condition (2) is always satisfied so the value of information is always positive. When \( \frac{\sigma_2^2}{2} = \kappa \gamma_t \), then \( \gamma_t = -\left( \rho \sigma_2 - \frac{\gamma}{\sigma_1} \right)^2 \) so the variance has a negative slope, which implies that \( \gamma_t = \frac{\sigma_2^2}{2\kappa} \) must be above \( \gamma \). This means that if the initial condition is such \( \overline{\gamma} \leq \gamma_t \leq \frac{\sigma_2^2}{2\kappa} \), the variance \( \gamma \) is decreasing overtime down to \( \overline{\gamma} \) and therefore condition (2) is always satisfied. Finally, if the initial condition is such that \( \frac{\sigma_2^2}{2\kappa} \leq \gamma_t \) the variance \( \gamma \) is decreasing overtime down to \( \overline{\gamma} \) and therefore condition (2) is not satisfied until time \( \tau \) defined by \( \gamma_\tau = \frac{\sigma_2^2}{2\kappa} \). We conclude that if the variance of initial beliefs is high enough, the value of information can be negative for a while before turning and remaining positive, assuming that the expiration date \( T \) is large enough. The direct effect of information may be negative.
4 Conclusion

We have explored the implications of incomplete information on the valuation of contingent claims, possibly when information does not only come from securities histories. Such frameworks apply to many real life situations, in particular in markets with weak monitoring institutions where the accuracy of the information may be low. Model 1 mainly applies for contracts on commodity when a convenience yield may not be observable by investors. Model 2 both applies for contracts on commodity and contact on financial assets. We use two different approaches to determine the fair price of a European call option and futures. In model 1, we compute the Radon-Nikodym derivative of the investor belief with respect to a fixed probability. For model 2, we use the general property that under incomplete information, the fair price of a European contract is equal to the expectation under agents’ beliefs of the price that would prevail under perfect information. In both models, when beliefs are normally distributed, we show that the variance of beliefs enhances the value of the call and the futures contract. In addition, the stock price volatility has two antagonistic effects. As in the case of complete information, it rises the value of the contract, but in addition the volatility damages the quality of the signal received, thus reducing learning and its associated value. Overall, the higher the volatility the greater the value of the option. Finally, we examine the case where the non-observable fundamental is a stochastic process. The main differences are that in general, beliefs no longer follow a martingale under the investor information structure and the investor’s Bayesian estimate may not converge to the true process.

In this paper, we have restricted our attention to the case of European contracts. It will be interesting but certainly a lot more difficult to investigate the effects of incomplete information on American contingent claims. This is left for future research.
5 Appendix

5.1 Appendix 1

**Proof.** Let $P(\mu)$ denote the price of a contract when the parameter $\mu$ is known. When agents cannot observe the true value of $\mu$ and have normally distributed beliefs $N(m, \gamma)$ about $\mu$, then the price of the contract is

$$Q(m, \gamma) = \frac{1}{\sqrt{2\pi}} \int \limits_\mathbb{R} P(x) e^{-\frac{(x-m)^2}{2\gamma}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \limits_\mathbb{R} P(\sqrt{\gamma}y + m)e^{-\frac{y^2}{2}} \, dy.$$

It follows that

$$Q_{mm}(m, \gamma) = \frac{1}{\sqrt{2\pi}} \int \limits_\mathbb{R} P''(\sqrt{\gamma}y + m)e^{-\frac{y^2}{2}} \, dy$$

$$Q_{\gamma}(m, \gamma) = \frac{1}{2\sqrt{2\pi}} \int \limits_\mathbb{R} P'(\sqrt{\gamma}y + m)ye^{-\frac{y^2}{2}} \, dy$$

$$= \frac{1}{2\sqrt{2\pi}} \left[ -P'(\sqrt{\gamma}y + m)e^{-\frac{y^2}{2}} \right]_{-\infty}^{\infty} + \frac{1}{2\sqrt{2\pi}} \int \limits_\mathbb{R} P''(\sqrt{\gamma}y + m)e^{-\frac{y^2}{2}} \, dy$$

$$= \frac{1}{2} Q_{mm}(m, \gamma) \text{ by assumption A2.}$$

5.2 Appendix 2

**Proof.** Under the probability measure $P_\delta$ the Radon-Nikodym derivative $\xi_\delta$ satisfies

$$d\xi_\delta(t) = -\xi_\delta(t) \frac{\Delta(t) - \delta}{\sigma_1(1 - \rho^2)} (dw_1(t) - \rho dw_2(t)).$$

Writing $\xi_\delta(t) = \xi(\Delta(t), \gamma(t))$ and using Ito lemma leads to

$$d\xi(t) = \xi_1 d\Delta(t) + \xi_2 \gamma(t) dt + \frac{1}{2} \xi_{11} \frac{\gamma^2(t)}{\sigma_1^2(1 - \rho^2)} dt$$

$$= \left( -\frac{\gamma^2(t)}{\sigma_1^2(1 - \rho^2)} \frac{1}{2} \xi_{11} - \xi_2 - \frac{\gamma(t)(\Delta(t) - \delta)}{\sigma_1^2(1 - \rho^2)} \xi_1 \right) dt - \frac{\gamma(t)}{\sigma_1(1 - \rho^2)} \xi_1 (dw_1(t) - \rho dw_2(t))$$

This implies

$$\xi_1(\Delta, \gamma) \frac{\gamma}{\sigma_1} = -\xi(\Delta, \gamma) \left( \frac{\delta - \Delta}{\sigma_1} \right)$$

$$\frac{1}{2} \xi_{11}(\Delta, \gamma) \frac{\gamma^2}{\sigma_1^2(1 - \rho^2)} - \xi_1(\Delta, \gamma) \frac{\gamma(\Delta - \delta)}{\sigma_1^2(1 - \rho^2)} - \xi_2(\Delta, \gamma) \frac{\gamma^2}{\sigma_1^2(1 - \rho^2)} = 0$$

The general solution to the first equation is

$$\xi(\Delta, \gamma) = f(\gamma) \exp \left( \frac{1}{\gamma} \left( \frac{\Delta^2}{2} - \delta \Delta \right) \right).$$
Plugging this expression into the second equation and after simplification shows that \( f \) must satisfy

\[
\frac{f'(\gamma)}{f(\gamma)} = \frac{1}{2\gamma} - \frac{\delta^2}{2\gamma^2}.
\]

Hence, \( f(\gamma) = A\sqrt{\gamma} \exp\left(\frac{\delta^2}{2\gamma}\right) \). Since \( \xi(0) = 1 \), we obtain

\[
\xi_\delta(t) = \sqrt{\frac{\gamma(t)}{\gamma_0}} \exp\left(\frac{1}{2\gamma(t)} (\Delta(t) - \delta)^2 - \frac{1}{2\gamma_0} (\Delta_0 - \delta)^2\right),
\]

which is the desired result. \( \square \)

### 5.3 Appendix 3

**Proof.** Define \( x(t) = \frac{\Delta(t) - \delta}{\gamma(t)} \). Then under the probability measure \( \mathcal{P}_\delta \), by Ito lemma we have

\[
dx(t) = \frac{d\Delta(t)}{\gamma(t)} - \frac{\dot{\gamma}(t)}{\gamma^2(t)} (\Delta(t) - \delta) dt
\]

\[
= \frac{1}{\sigma_1(1-\rho^2)} \left(-dw_1(t) - \frac{1}{\sigma_1} (\Delta(t) - \delta) dt + \rho dw_2(t)\right) + \frac{\Delta(t) - \delta}{\sigma_1^2(1-\rho^2)} dt
\]

\[
= \frac{1}{\sigma_1(1-\rho^2)} (-dw_1(t) + \rho dw_2(t)).
\]

It follows that the value of the call \( C \) can be written

\[
C(S_t, \Delta_t, \gamma_t, t) = \sqrt{\frac{\gamma(T)}{\gamma_t}} e^{-\frac{\gamma(t)}{2} x^2(t) - r(T-t)} E^\mathcal{F}_t \left[ e^\frac{\gamma(T)}{2} x^2(T) \left[ S(T) - K \right]^+ \right]
\]

Define

\[
d_2 = \frac{\ln(S_t/K) + (r - \delta - \frac{\sigma^2 t}{2}) (T-t)}{\sigma_1 \sqrt{T-t}}.
\]

Then

\[
E^\mathcal{F}_t \left[ e^\frac{\gamma(T)}{2} x^2(T) \left[ S(T) - K \right]^+ \right] = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-d_2}^{\infty} e^{-\frac{\gamma(T)}{2} x^2(T)} \left( x - \frac{\gamma(t)}{\gamma_t} x_1 + \frac{\gamma(t)}{\gamma_t} \sqrt{1-\rho^2} y \right)^2 \times
\]

\[
\left( S_t e^{(r-\delta-\frac{\sigma^2 t}{2})(T-t)+\sigma_1 \sqrt{T-t} u - K} \right) e^{-\frac{y^2}{2} e^{-\frac{y^2}{2}}} dy.
\]

Define

\[
I_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-d_2}^{\infty} e^{-\frac{\gamma(T)}{2} x^2(T)} \left( x - \frac{\gamma(t)}{\gamma_t} x_1 + \frac{\gamma(t)}{\gamma_t} \sqrt{1-\rho^2} y \right)^2 e^{-\frac{y^2}{2}} e^{-\frac{y^2}{2}} dy
\]

\[
I_1 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-d_2}^{\infty} e^{-\frac{\gamma(T)}{2} x^2(T)} \left( x - \frac{\gamma(t)}{\gamma_t} x_1 + \frac{\gamma(t)}{\gamma_t} \sqrt{1-\rho^2} y \right)^2 \left( e^{(r-\delta-\frac{\sigma^2 t}{2})(T-t)+\sigma_1 \sqrt{T-t} u} \right) e^{-\frac{y^2}{2}} e^{-\frac{y^2}{2}} dy
\]
Setting $A(u) = x_t - \frac{\sqrt{T-t}}{\sigma_1} u$ it follows that

\[
I_2 = \frac{1}{2\pi} \int_{-d_2}^{\infty} e^{-\frac{1}{2} \left( y^2 \gamma(T) \left( \frac{1}{\gamma_t} + \frac{t-x_t}{\sigma_1^2} \right) - 2 \frac{\gamma(T) \sqrt{T-t}}{\sigma_1} \frac{e}{\sqrt{1-\rho^2}} A(u) y \right) + \frac{\gamma(T) A^2(u)}{2}} \, dy e^{-\frac{u^2}{2}} \, du
\]

\[
= \frac{1}{\sqrt{\gamma(T) \left( \frac{1}{\gamma_t} + \frac{T-x_t}{\sigma_1^2} \right) 2\pi}} \int_{d_2}^{\infty} e^{-\frac{1}{2} \left( z + \frac{1}{\sqrt{1 + \frac{T-t}{T}}} \frac{\gamma(T) \sqrt{T-t}}{\sqrt{1-\rho^2}} \right)^2} \, dz e^{-\frac{u^2}{2}} \, du
\]

\[
= \frac{1}{\sqrt{\gamma(T) \left( \frac{1}{\gamma_t} + \frac{T-t}{\sigma_1^2} \right) \sqrt{2\pi}}} \int_{d_2}^{\infty} e^{-\frac{1}{2} \left( \frac{\gamma_t \sqrt{T-t} x_t}{\sigma_1} + \frac{\gamma \sqrt{T-t} u}{\sigma_1^2} \right)^2 + \frac{\gamma t^2}{\sigma_1^2}} \, du
\]

\[
= \frac{\sqrt{\gamma_t}}{\sqrt{\gamma(T)} \sqrt{2\pi}} \frac{1}{\sqrt{1 + \gamma t (T-t)}} \int_{d_2}^{\infty} e^{-\frac{1}{2} \left( s + \frac{\gamma_t \sqrt{T-t} x_t}{\sigma_1} \right)^2 + \frac{\gamma t^2}{\sigma_1^2}} \, ds
\]

Then, define

\[
d_2' = \frac{d_2}{\sqrt{1 + \gamma t (T-t)}} - \frac{\gamma t \sqrt{T-t} x_t}{\sigma_1 \sqrt{1 + \gamma t (T-t)}}
\]

\[
= \frac{\ln(S_t) + \left( r - \Delta_t - \frac{\sigma_t^2}{2} \right) (T-t)}{\sigma_1 \sqrt{T-t} \sqrt{1 + \gamma t (T-t)}}.
\]

It follows that

\[
I_2 = \frac{\gamma t}{\gamma(T) \sqrt{2\pi}} \int_{-d_2'} e^{-\frac{z^2}{2}} \, dz
\]

\[
= \frac{\gamma t}{\gamma(T)} e^{-\frac{x_t^2}{2}} N(d_2').
\]

In the same fashion,

\[
I_1 = \frac{1}{\sqrt{\gamma(T) \left( \frac{1}{\gamma_t} + \frac{T-t}{\sigma_1^2} \right) \sqrt{2\pi}}} \int_{-d_2}^{\infty} e^{-\frac{1}{2} \left( \frac{(x_t - \frac{\sqrt{T-t}}{\sigma_1} u)^2}{2 \left( \frac{1}{\gamma_t} + \frac{T-t}{\sigma_1^2} \right)} \right)} e^{(r - \delta - \frac{x_t^2}{2}) (T-t)} + \sigma_1 \sqrt{T-t} u e^{-\frac{u^2}{2}} \, du
\]

\[
= \frac{1}{\sqrt{\gamma(T) \left( \frac{1}{\gamma_t} + \frac{T-t}{\sigma_1^2} \right) \sqrt{2\pi}}} \int_{-d_2}^{\infty} e^{-\frac{1}{2} \left( \frac{\gamma_t \sqrt{T-t} x_t}{\sigma_1} + \frac{\gamma \sqrt{T-t} u}{\sigma_1^2} \right)^2 + \frac{\gamma t^2}{\sigma_1^2}} \, du
\]
Then, set

\[ d'_1 = d'_2 + \frac{\gamma_t \sigma_1 \sqrt{T-t}}{\sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}} \left( \frac{1}{\gamma_t} + \frac{T-t}{\sigma_1^2} \right) \]

\[ = \ln \left( \frac{S_t}{K} \right) + \left( r - \Delta_t + \gamma_t (T-t) + \frac{\sigma_1^2}{2} \right) (T-t) \]

\[ \frac{\ln(\frac{S_t}{K}) + \left( r - \Delta_t + \gamma_t (T-t) + \frac{\sigma_1^2}{2} \right) (T-t)}{\sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}} \]

It follows that

\[ I_1 = \sqrt{\frac{\gamma_t}{\gamma(T)}} e^{\frac{\gamma_t^2}{2} + (r-\Delta_t)(T-t) + \frac{\gamma_t^2}{2} (T-t)^2} \frac{1}{\sqrt{2\pi}} \int_{-d'_1}^{\infty} e^{-\frac{z^2}{2}} dz \]

\[ = \sqrt{\frac{\gamma_t}{\gamma(T)}} e^{\frac{\gamma_t^2}{2} + (r-\Delta_t)(T-t) + \frac{\gamma_t^2}{2} (T-t)^2} N(d'_1). \]

Finally, we obtain that

\[ C(S_t, \Delta_t, \gamma_t, t) = e^{-\Delta_t(T-t) + \frac{\gamma_t^2}{2} (T-t)^2} S_t N(d'_1) - e^{-r(T-t)} K N(d'_2), \]

which is the desired result. □

5.4 Appendix 4

Proof. Delta Since \( d'_1 = d'_2 + \sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}} \), it follows that

\[ \frac{\partial d'_1}{\partial S_t} = \frac{\partial d'_2}{\partial S_t}. \]

Then

\[ \frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial S_t} = e^{-\Delta_t(T-t) + \frac{\gamma_t^2}{2} (T-t)^2} N(d'_1) + e^{-r(T-t)} \left( S_t e^{(r-\Delta_t)(T-t) + \frac{\gamma_t^2}{2} (T-t)^2} \left( n(d'_1) \frac{\partial d'_1}{\partial S_t} - n(d'_2) \frac{\partial d'_2}{\partial S_t} \right) - K n(d'_2) \frac{\partial d'_2}{\partial S_t} \right), \]

where \( n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \). Moreover, since \( d'_1 = d'_2 + \sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}} \), it follows that

\[ n(d'_1) = n(d'_2) \frac{K}{S_t} e^{-\gamma_t(t)(T-t) - \frac{\gamma_t^2}{2} (T-t)^2}. \]

Hence

\[ \frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial S_t} = e^{-\Delta_t(T-t) + \frac{\gamma_t^2}{2} (T-t)^2} N(d'_1) > 0. \]
Gamma We have seen that 
\[ \frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial S_t} = e^{-\Delta_t (T-t) + \frac{\gamma_t}{2} (T-t)^2} \frac{\partial 
}{\partial S_t} \]
\[ \frac{\partial^2 C(S_t, \Delta_t, \gamma_t, t)}{\partial S_t^2} = e^{-\Delta_t (T-t) + \frac{\gamma_t}{2} (T-t)^2} \frac{\partial n(d'_1)}{\partial S_t} \]
\[ = \frac{e^{-\Delta_t (T-t) + \frac{\gamma_t}{2} (T-t)^2} n(d'_1)}{\sigma_1 S_t \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}} > 0. \]

Theta Since \( d'_1 = d'_2 + \sqrt{\sigma_1^2 - \gamma_t^2}, \) it follows that
\[ \frac{\partial d'_2}{\partial \tau} = \frac{\partial d'_2}{\partial \tau} + \frac{\sigma_1^2 + 2 \gamma_t \tau}{2 \sqrt{\sigma_1^2 + \gamma_t (T-t)^2}}. \]

Therefore
\[ \frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial \tau} = (-\Delta_t + \tau \gamma_t) e^{-\Delta_t \tau + \frac{\gamma_t}{2} \tau^2} S_t N(d'_1) + r e^{-\tau} K N(d'_2) \]
\[ + \frac{e^{-\tau}}{S_t e^{(r-\Delta_t) \tau + \frac{\gamma_t}{2} \tau^2}} \left( n(d'_1) \frac{\partial d'_1}{\partial \tau} - K n(d'_2) \frac{\partial d'_2}{\partial \tau} \right) \]
\[ = (-\Delta_t + \tau \gamma_t) e^{-\Delta_t \tau + \frac{\gamma_t}{2} \tau^2} S_t N(d'_1) + r e^{-\tau} K N(d'_2) \]
\[ + \frac{S_t e^{-\Delta_t \tau + \frac{\gamma_t}{2} \tau^2} n(d'_1)}{2 \sqrt{\sigma_1^2 + \gamma_t (T-t)^2}} \frac{\sigma_1^2 + 2 \gamma_t \tau}{\sqrt{\sigma_1^2 + \gamma_t (T-t)^2}}. \]

Vega Since \( d'_1 = d'_2 + \sqrt{T-t} \sqrt{\sigma_1^2 + \gamma_t (T-t)^2}, \) it follows that
\[ \frac{\partial d'_2}{\partial \sigma_1^2} = \frac{\partial d'_2}{\partial \sigma_1^2} + \frac{(T-t)}{2 \sqrt{\sigma_1^2 + \gamma_t (T-t)^2}}. \]

Then
\[ \frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial \sigma_1^2} = e^{-r(T-t)} \left( S_t e^{(r-\Delta_t)(T-t) + \frac{\gamma_t}{2} (T-t)^2} n(d'_1) \frac{\partial d'_1}{\partial \sigma_1^2} - K n(d'_2) \frac{\partial d'_2}{\partial \sigma_1^2} \right). \]

Since \( n(d'_1) = n(d'_2) K e^{(r-\Delta_t)(T-t) - \frac{\gamma_t}{2} (T-t)^2}, \) it follows that
\[ \frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial \sigma_1^2} = S_t e^{-\Delta_t (T-t) + \frac{\gamma_t}{2} (T-t)^2} \frac{(T-t)}{2 \sqrt{\sigma_1^2 + \gamma_t (T-t)^2}} n(d'_1) > 0. \]

Rho
\[ \frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial \rho} = (T-t) e^{-r(T-t)} K N(d'_2) > 0. \]

5.5 Appendix 5

Learning Effect: In order to properly disentangle the effect of \( \sigma_1 \) on the learning value, we write
\[ d'_2(a) = \frac{\ln \left( \frac{S_t}{K} \right) + \left( r - \Delta_t - \frac{\sigma_1^2}{2} \right) (T-t)}{\sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}} \]
\[ d'_1(a) = d'_2(a) + \sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}, \]
and we investigate the effect of a change in the parameter \( a \) on the price \( C \). We obtain

\[
\frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial a} = e^{-r(T-t)} \left( S_t e^{(r-\Delta_t)(T-t)+\frac{\gamma_t}{2}(T-t)^2} n(d'_1) \frac{\partial d'_1}{\partial a} - Kn(d'_2) \frac{\partial d'_2}{\partial a} \right)
\]

\[
= - \left( S_t e^{-\Delta_t(T-t)+\frac{\gamma_t}{2}(T-t)^2} n(d'_1) \frac{\gamma_t(T-t)}{2\sigma_1 a \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{a}}} \right) < 0.
\]

**Proof of Proposition 4.** Since \( d'_1 = d'_2 + \sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}} \), it follows that

\[
\frac{\partial d'_1}{\partial \gamma_t} = \frac{\partial d'_2}{\partial \gamma_t} + \frac{(T-t) \sqrt{T-t}}{2\sigma_1 \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}}
\]

Then

\[
\frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial \gamma_t} = e^{-r(T-t)} \left( S_t e^{(r-\Delta_t)(T-t)+\frac{\gamma_t}{2}(T-t)^2} \left( \frac{(T-t)^2}{2} N(d'_1) + n(d'_1) \frac{\partial d'_1}{\partial \gamma_t} \right) - Kn(d'_2) \frac{\partial d'_2}{\partial \gamma_t} \right),
\]

where \( n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \). Moreover, since \( d'_1 = d'_2 + \sigma_1 \sqrt{T-t} \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}} \), it follows that

\[
n(d'_1) = n(d'_2) \frac{K}{S_t} e^{-(r-\Delta_t)(T-t)-\frac{\gamma_t}{2}(T-t)^2}.
\]

Hence

\[
\frac{\partial C(S_t, \Delta_t, \gamma_t, t)}{\partial \gamma_t} = S_t e^{-\Delta_t(T-t)+\frac{\gamma_t}{2}(T-t)^2} \left( \frac{(T-t)^2}{2} N(d'_1) + n(d'_1) \frac{(T-t) \sqrt{T-t}}{2\sigma_1 \sqrt{1 + \gamma_t \frac{T-t}{\sigma_1^2}}} \right) > 0.
\]

**5.6 Appendix 6**

**Proof.** European Call Option: Gibson and Schwartz Model (1990) under perfect information. Integrating the law of motion of the convenience yield leads to

\[
\delta(s) = (1 - e^{-\kappa(s-t)})\alpha + e^{-\kappa(s-t)} \delta(t) + \sigma_2 \int_t^s e^{-\kappa(s-u)} dw_2(u).
\]

It follows that

\[
S(T)e^{-r(T-t)} = e^{m(\delta_t; T-t; \alpha) + \sigma_1 (w_1(T)-w_1(t)) - \sigma_2 \int_t^T e^{-\kappa(s-u)} dw_2(u) ds}
\]

where

\[
m(\delta_t; T-t; \alpha) = \frac{\sigma_1^2}{2} (T-t) - \frac{1 - e^{-\kappa(T-t)}}{\kappa} \delta_t - \left( T - t - \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right) \alpha.
\]
The stochastic process

\[ X(t, T) = \sigma_1(w_1(T) - w_1(t)) - \sigma_2 \int_t^T \int_t^s e^{-\kappa(s-u)} dw_2(u) ds \]

is normally distributed with mean equal to zero and variance

\[ \Sigma(T-t) = \sigma_1^2(T-t) + \sigma_2^2 \int_0^{T-t} \left( 1 - \frac{e^{-\kappa(T-t-s)}}{\kappa} \right)^2 ds - \frac{2\rho\sigma_1\sigma_2}{\kappa} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right). \]

Define

\[ d_1 = d_2 + \sqrt{\Sigma(T-t)} \]

\[ d_2 = \frac{\ln\left( \frac{\tilde{S}_t}{K} \right) + m(\delta_t, T-t; \alpha)}{\sqrt{\Sigma(T-t)}}, \]

and it follows that

\[ C(S_t, \delta_t, t) = e^{m(\delta_t, T-t; \alpha) + \frac{\Sigma(T-t)}{2} S_t N(d_1) - e^{-r(T-t)} KN(d_2)}. \]

The proof is complete. ■

**Comparative Statics**

Shadow price of information \( \frac{\partial C}{\partial \gamma_t} \). **Proof.** Since \( d'_1 = d'_2 + \sqrt{\Sigma(T-t)} \sqrt{\Omega(T-t)} \), it follows that

\[ \frac{\partial d'_1}{\partial \gamma_t} = \frac{\partial d'_2}{\partial \gamma_t} + \frac{\left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2}{2\sqrt{\Sigma(T-t) + \gamma_t \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2}}. \]

Then

\[ C_{\gamma} = \frac{1}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 e^{m(\delta_t, T-t; \alpha) + \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}} S_t N(d'_1) + e^{m(\delta_t, T-t; \alpha) + \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}} S_t n(d'_1) \frac{\partial d'_1}{\partial \gamma_t} - e^{-r(T-t)} KN(d'_2) \frac{\partial d'_2}{\partial \gamma_t}, \]

where \( n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \). Moreover, since \( d'_1 = d'_2 + \sqrt{\Sigma(T-t)} \sqrt{\Omega(T-t)} \), it follows that

\[ n(d'_1) = n(d'_2) \frac{K}{S_t} e^{-m(\delta_t, T-t; \alpha) + \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2} - r(T-t)}. \]

Hence

\[ C_{\gamma} = \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 S_t e^{m(\delta_t, T-t; \alpha) + \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}} \left( N(d'_1) + \frac{n(d'_1)}{\sqrt{\Sigma(T-t) \sqrt{\Omega(T-t)}}} \right) > 0. \]
Spot price volatility effect $\frac{\partial C}{\partial \sigma^2_1}$. \textbf{Proof.} Since $d_1' = d_2' + \sqrt{\Sigma(T-t)}\sqrt{\Omega(T-t)}$, it follows that

$$\frac{\partial d_1'}{\partial \sigma^2_1} = \frac{\partial d_2'}{\partial \sigma^2_1} + \frac{(T-t) - \frac{\rho \sigma_2}{\kappa \sigma_1} (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})}{2\sqrt{\Sigma(T-t) + \gamma_t (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})^2}}.$$

Then

$$C_{\sigma^2_1} = \frac{1}{2} \left( T-t - \frac{\rho \sigma_2}{\kappa \sigma_1} (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa}) \right) \times \frac{e^{m(\delta_t,T-t;\alpha_t) + \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}}}{\sqrt{\Sigma(T-t) + \gamma_t (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})^2}} S_t N(d_1'),$$

where $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Moreover, since

$$n(d_1') = n(d_2') \frac{K}{S_t} e^{-m(\delta_t,T-t;\alpha_t) - \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 - \frac{\Sigma(T-t)}{2} - r(T-t)},$$

it follows that

$$C_{\sigma^2_1} = \frac{1}{2} \left( T-t - \frac{\rho \sigma_2}{\kappa \sigma_1} (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa}) \right) S_t e^{m(\delta_t,T-t;\alpha_t) + \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}} \left( N(d_1') + \frac{n(d_1')}{\sqrt{\Sigma(T-t) + \gamma_t (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})^2}} \right).$$

It follows that $C_{\sigma^2_1}$ is positive exactly when $T-t - \frac{\rho \sigma_2}{\kappa \sigma_1} (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})$ is positive, which is always the case when $\rho$ is small. \hfill \blacksquare

Convenience yield/dividend rate volatility effect $\frac{\partial C}{\partial \sigma^2_2}$. \textbf{Proof.} Since $d_1' = d_2' + \sqrt{\Sigma(T-t)}\sqrt{\Omega(T-t)}$, it follows that

$$\frac{\partial d_1'}{\partial \sigma^2_2} = \frac{\partial d_2'}{\partial \sigma^2_2} + \frac{\int_0^{T-t} \left( 1-e^{-\kappa(T-s-t)} \right)^2 ds - \frac{\rho \sigma_1}{\kappa \sigma_2} (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})}{2\sqrt{\Sigma(T-t) + \gamma_t (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})^2}}.$$

Thus

$$C_{\sigma^2_2} = \frac{1}{2} \left( \int_0^{T-t} \left( 1-e^{-\kappa(T-t-s)} \right)^2 ds - \frac{\rho \sigma_1}{\kappa \sigma_2} (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa}) \right) \times S_t e^{m(\delta_t,T-t;\alpha_t) + \frac{\gamma_t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}} \left( N(d_1') + \frac{n(d_1')}{\sqrt{\Sigma(T-t) + \gamma_t (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})^2}} \right).$$

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It follows that $C_{\sigma_2}$ is positive exactly when $\int_0^{T-t} \left( \frac{1-e^{-\kappa(T-t-s)}}{\kappa} \right)^2 ds - \frac{\rho a_1}{\kappa \sigma_2} (T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa})$ is positive, which is always the case when $\rho$ is small.

Learning Effect. In order to properly disentangle the effect of $\sigma_2$ on the learning value, we write

\[ d_1'(a,b) = d_2'(a,b) + \sqrt{\Sigma(T-t)} \sqrt{\Omega(T-t,a,b)} \]

\[ d_2'(a,b) = \frac{\ln(S_t) + m(\delta_t, T-t; \alpha)}{\sqrt{\Sigma(T-t)} \sqrt{\Omega(T-t, a, b)}}. \]

where

\[ \Omega(T-t, a, b) = 1 + \frac{\gamma t}{\Sigma(T-t, a, b)} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 \]

\[ \Sigma(T-t, a, b) = a^2(T-t) + b^2 \int_0^{T-t} \left( \frac{1-e^{-\kappa(T-t-s)}}{\kappa} \right)^2 ds - \frac{2 \rho ab}{\kappa} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right) \]

and we investigate the effect of a change in the parameter $a$ on the price $C$. We obtain

\[ C_a = \left( m(\delta_t, T-t; \alpha_1) + \frac{\gamma t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2} S_t n(d'_1) \frac{\partial d'_1}{\partial a} - e^{-r(T-t)} K n(d'_2) \frac{\partial d'_2}{\partial a} \right) \]

\[ = -2e^{m(\delta_t, T-t; \alpha_1)} + \frac{\gamma t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2} S_t \left( \frac{\gamma t \sqrt{\Sigma(T-t)} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 n(d'_1)}{2 \sqrt{\Omega(t-a, b) \Sigma(T-t, a, b)}} \times \right) \]

\[ a(T-t) - \frac{\rho b}{\kappa} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right) \].

We see that if $\rho$ is small enough, an increase in $\sigma_1$ will decrease the value associated with learning. In a similar fashion

\[ C_b = \left( e^{m(\delta_t, T-t; \alpha_1)} + \frac{\gamma t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2} S_t n(d'_1) \frac{\partial d'_1}{\partial b} - e^{-r(T-t)} K n(d'_2) \frac{\partial d'_2}{\partial b} \right) \]

\[ = -2e^{m(\delta_t, T-t; \alpha_1)} + \frac{\gamma t}{2} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2} S_t \left( \frac{\gamma t \sqrt{\Sigma(T-t)} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 n(d'_1)}{2 \sqrt{\Omega(t-a, b) \Sigma(T-t, a, b)}} \times \right) \]

\[ b \int_0^{T-t} \left( \frac{1-e^{-\kappa(T-t-s)}}{\kappa} \right)^2 ds - \frac{\rho a}{\kappa} \left( T-t - \frac{1-e^{-\kappa(T-t)}}{\kappa} \right) \].

The conclusion is the same if $\rho$ is small enough, an increase in $\sigma_2$ will decrease the value associated to learning.
5.7 Appendix 7

Shadow price of information \( \frac{\partial C}{\partial \gamma} \). **Proof.** Since \( d'_1 = d'_2 + \sqrt{\Sigma(T-t)}V(T-t) \), it follows that

\[
\frac{\partial d'_1}{\partial \gamma} = \frac{\partial d'_2}{\partial \gamma} + \frac{\left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2}{2\sqrt{\Sigma(T-t) + \gamma t} \left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2}.
\]

Then

\[
C_\gamma = \frac{1}{2} \left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 e^{m(\delta_t, T-t; \alpha_t) + \frac{\gamma t}{2} \left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}} S_t N(d'_1) +
\]

\[
e^{-m(\delta_t, T-t; \alpha_t) + \frac{\gamma t}{2} \left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2} S_t n(d'_1) \frac{\partial d'_1}{\partial \gamma} - e^{-r(T-t)} K n(d'_2) \frac{\partial d'_2}{\partial \gamma},
\]

where \( n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \). Moreover, since \( d'_1 = d'_2 + \sqrt{\Sigma(T-t)}V(T-t) \), it follows that

\[
n(d'_1) = n(d'_2) \frac{K}{S_t} e^{-m(\delta_t, T-t; \alpha_t) - \frac{\gamma t}{2} \left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 - \frac{\Sigma(T-t)}{2} - r(T-t)}.
\]

Hence

\[
C_\gamma = \left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 S_t e^{m(\delta_t, T-t; \alpha_t) + \frac{\gamma t}{2} \left( \frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 + \frac{\Sigma(T-t)}{2}} \left( N(d'_1) + \frac{n(d'_1)}{\sqrt{\Sigma(T-t)V(T-t)}} \right) > 0. \]

\[\blacksquare\]
6 References


