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Stochastic growth: a duality approach

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Abstract

We re-examine the representative agent's optimal consumption and savings under uncertainty in the presence of investment constraints using martingale representation and convex analysis techniques. This framework allows us to explicitly quantify precautionary savings which induces a higher average growth rate than in a certainty setup. We provide a closed form solution for a Cobb–Douglas economy. The effect of uncertainty on portfolio selection is analyzed. Consumption growth rate and risk free interest rate exhibit a U-shaped relationship. Uncertainty negatively affects expected consumption growth rate; such a result seems to be supported by empirical evidence.

JEL classification: C61; D90; O41

Keywords: Dual problem; Precautionary savings; Stochastic growth

1. Introduction

We re-examine the representative agent's optimal consumption and savings under uncertainty in the presence of investment constraints. Levhari and Srinivasan [10], Brock and Mirman [3], and Levhari [11], using dynamic programming techniques (DPT) to derive the necessary conditions that optimal policies need to satisfy, are among the pioneers to study the properties of the optimal consumption and investment policy functions within a discrete-time, neoclassical one-sector model. In a continuous time framework, Bourguignon [2] and Merton [13] compute the steady-state distributions of the capital–labor ratio, interest rate, per capita consumption

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and output, etc. Obstfeld [14] studies the impact of risk diversification on economic growth. All these authors use the primal approach that leads to solving a highly non-linear Hamilton Jacobi Bellman equation, which in few cases can be explicitly worked out with a lucky guess. An alternative approach to DPT is the work by Bismut [1] who uses extended Hamiltonian techniques to deal with intertemporal optimal allocation in presence of risks.

We derive results using a duality approach in the presence of constraints that relies on martingale representation and convex analysis techniques similar to those exposed in [4–6]. In particular, it is worth noting that the methodology applies for incomplete markets. The dual program turns out to be much easier to analyze than the primal program. For a general dynamic framework, we are able to explicitly quantify precautionary savings as a function of the relative risk aversion and relative prudence ratios, generalizing the Drèze-Modigliani [7] "substitution effect" and Kimball's results [9]. We show that investment constraints reduce precautionary savings and thus the consumption growth rate. When the technology is Cobb–Douglas and preferences exhibit CRRA, the portfolio analysis confirms the Levhari and Srinivasan [10] conjecture according to which "when the variance of an asset is increased keeping its mean constant, the optimal proportion invested in this asset goes down." Moreover, the savings rate responds positively to an increase in uncertainty exactly when the coefficient of relative risk aversion is above unity, reflecting consumption smoothing. The expected consumption growth rate and the risk-free interest rate exhibit a U-shaped relationship. As the risky technology becomes more volatile or the individual is more risk averse, the risk-free technology tends to be preferred. We extend the analysis to the autarky case when only a risky technology is available.

2. The general economic setting

We consider a continuous-time economy with a finite horizon T in which a representative agent has to choose an optimal consumption and investment policy.

2.1. The Economy

Information structure. Uncertainty is modeled by a probability space $(\Omega, \mathscr{F}, P)$ on which is defined an n-dimensional (standard) Brownian motion w. A state of nature ω is an element of Ω. $\mathscr{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P is assigned. Let $\mathscr{F}_t$ be the σ-algebra generated by the observations of w and augmented. The filtration $\mathbb{F} = \{\mathscr{F}_t, t \in [0, T]\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented).

Individual preferences. There is a single perishable good available for consumption, the numéraire. Preferences are represented by a time additive utility function

$$U(c) = E\left[\int_0^T u(c(t))e^{-\theta t}\, dt\right],$$

where the instantaneous utility function u is twice continuously differentiable, increasing and strictly concave and θ denotes the subjective discount rate of future. In addition, u satisfies the following Inada conditions: $\lim_{c\to 0^+} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$.

The financial market and the technology. The financial market consists only in a locally risk-free bond whose price B evolves according to the following equation $dB(t) = r(t)B(t)dt$, where r is the international interest rate. Alternatively, one may interpret the risk-free asset as a linear storage technology. The real sector is modeled by a production function $F : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that uses n different types of inputs k_i and is affected by a shock a. F is assumed to be increasing and jointly concave in its n first arguments. In the sequel, $k = (k_1, \ldots, k_n)$ (respectively $K = (K_1, \ldots, K_n)$) denotes the vector of quantity (respectively value) of capital inputs, with $K_i = p_i k_i$. Input prices p and the real shock a are assumed to be non-negative Ito processes.

The total wealth W is the sum of the value of inputs and the value X invested into the bond, i.e. $W = K^\top \bar{1} + X$. The economy starts with some positive wealth $W_0 > 0$.

Investment constraints. Input quantities must be positive. More generally, we require (X, K) to be in a *convex closed* set $Q \subseteq \mathbb{R}^{n+1}$. When borrowing is prevented, $Q = \mathbb{R}_+ \times \mathbb{R}^n$; when borrowing is allowed up to a maximum fraction $M > 0$ of the wealth, $Q = \{(X, K) \in \mathbb{R} \times \mathbb{R}^n_+, \frac{-X 1_{\{X \leqslant 0\}}}{W} \leqslant M\}$[1] (maximum borrowing ratio).

Feasibility. A consumption plan c is feasible if there is a couple $(X, K) \in Q$ such that

$$dW(t) = (F(K(t)/p(t), a(t)) + r(t)X(t) - c(t))\,dt + K(t)^\top \sigma(t)\,dw(t),$$
$$W(t) \geqslant -\underline{W} \text{ and } W(T) \geqslant 0.$$

Short-term deficits up to a maximum $\underline{W} \geqslant 0$ are allowed but the country must end up with no debt. The diffusion term encapsulates differences between what can be foreseen, $E[dW(t) \mid \mathcal{F}_t]$ and what is actually realized $dW(t)$. It can be considered as a stochastic adjustment cost equal to zero on average. The linearity of the diffusion term is important both for *existence and tractability* reasons. Let $\mathscr{C}$ denote the set of feasible consumption plans.

Technical restrictions on the stochastic processes r, c, X, K and σ to ensure existence of a solution can be found in [3]. We now examine the representative agent problem.

2.2. The representative agent problem

The representative agent maximizes her expected discounted life time utility:

$$\begin{aligned}
&\max_{(c \in \mathscr{C}, (K,X) \in Q)} && E\left[\int_0^T u(c(t))e^{-\theta t}\,dt\right] \\
&\text{s.t.} && dW(t) = (F(k(t), a(t)) + r(t)X(t) - c(t))\,dt + K(t)^\top \sigma(t)\,dw(t) \\
& && W(t) \geqslant -\underline{W},\ W(T) \geqslant 0,\ W_0 \text{ given.}
\end{aligned} \tag{1}$$

[1] 1_A is the indicator function such that $1_A(a) = 1$ if $a \in A$ and 0 otherwise.

2.3. The dual approach

We use a dual approach to convert the (primal) *dynamic* problem into an equivalent (dual) *static* problem. The methodology relies on convex duality techniques developed by Cvitanic and Karatzas [5] and Cuoco and Cvitanic [4].

Effective domain. For $(v_0, v) \in \mathbb{R} \times \mathbb{R}^n$, define $e(v_0, v) = \sup_{(X,K)\in Q} F(K/p, a) - v_0 X - v^\top K$. The effective domain of F is $\mathcal{N} = \{(v_0, v) \in \mathbb{R} \times \mathbb{R}^n : \exists M \in \mathbb{R}, \forall (v_0, v), e(v_0, v) \leqslant M\}$. It is a closed convex set. $e(v_0, v)$ can be interpreted as an instantaneous profit. The effective domain is the set of pseudo price vectors (v_0, v) compatible with a finite profit.

Then, for $(v_0, v) \in \mathcal{N}$, define the exponential martingale

$$\xi_v(t) = \exp\left(\int_0^t -\frac{||\kappa_v(u)||^2}{2} du + \kappa_v(u)^\top \, dw(u)\right),$$

with[2] $\kappa_v(t) = -\sigma^{-1}(t)(v(t) - (r(t) + v_0(t)\bar{1})$, the discount factor

$$\beta_v(t) = \exp\left(\int_0^t -(r(u) + v_0(u)) \, du\right),$$

and $\pi_v(t) = \beta_v(t)\xi_v(t)$ which is interpreted in the sequel as a state price density.

Finally, let $\tilde{u}(y, t) = \max_{c \geqslant 0} u(c, t) - yc$ denote the convex conjugate of $u(c, t) = u(c)e^{-\theta t}$.

Under some concavity conditions which are satisfied here, it is enough to determine the saddle point $(c^*, \psi^*, (v_0^*, v^*))$ of the functional

$$\begin{aligned}\mathcal{L}(c, \psi, (v_0, v)) = {} & E\left[\int_0^T u(c(t))e^{-\theta t} \, dt\right] \\ & - \psi\left(E\left[\int_0^T \pi_v(t)(c(t) - e(v_0(t), v(t))) \, dt\right] - W_0\right)\end{aligned}$$

to solve the primal program. The maximization over c yields $u'(c^*(t))e^{-\theta t} = \psi\pi_v(t)$, and the dual program is:

$$\begin{aligned}\min_{(\psi,(v_0,v)) \in \mathbb{R}_{++} \times \mathcal{N}} \; & E\left[\int_0^T \tilde{u}(\psi\pi_v(t), t) \, dt\right] \\ + \psi E & \left[\int_0^T \pi_v(t) e(v_0(t), v(t)) \, dt\right] + \psi W_0. \quad \text{(DP)}\end{aligned}$$

In general, it is *not* possible to obtain an explicit solution of this program. However, the task is easier *when e equals zero*; it is in particular the case when the production function is *homogenous of degree one and $\mathcal{N}$ is a cone*. We now turn to the main result of the paper.

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[2] For X in $\mathbb{R}^n \times \mathbb{R}^m$, the notation $||X||^2$ refers to the norm of X and $||X||^2 = Tr(X^\top X)$.

Theorem 1. *Assume that* u''' *exists and is positive. Denoting* $AR(c) = -\frac{u''(c)}{u'(c)}$ $\left(RR(c) = -\frac{cu''(c)}{u'(c)}\right)$ *the absolute (relative) risk aversion ratio and* $AP(c) = -\frac{u'''(c)}{u''(c)}$ $\left(RP(c) = -\frac{cu'''(c)}{u''(c)}\right)$ *the absolute (relative) prudence ratio, the evolution the optimal consumption level* c^* *(expected growth rate of consumption* g^**) is governed by the absolute (relative) risk aversion and prudence ratios. In particular, in presence of uncertainty,* g^* *is higher than in the classical Ramsey model [16].*

Proof. The optimal condition is $u'(c^*(t))e^{-\theta t} = \psi \pi_{v^*}(t)$. Set $\varsigma_{v^*}(t) = \psi \pi_{v^*}(t) e^{\theta t}$ and $c^*(t) = (u')^{-1}(\varsigma_{v^*}(t))$. By Ito's lemma,

$$d\varsigma_{v^*}(t) = (u')^{-1}(c^*(t))((r(t) + v_0^*(t) - \theta)\,dt + \kappa_{v^*}(t)^\top\,dw(t)).$$

Then, since $\frac{d(u')^{-1}}{dx}(u'(x)) = \frac{1}{u''(x)}$ and $\frac{d^2(u')^{-1}}{dx^2}(u'(x)) = -\frac{u'''(x)}{(u''(x))^2}$, it follows that

$$\begin{aligned} dc^*(t) = &\frac{1}{AR(c^*(t))}[r(t) + v_0^*(t) - \theta]\,dt + \frac{1}{2}\frac{AP(c^*(t))}{(AR(c^*(t)))^2}||\kappa_{v^*}(t)||^2\,dt \\ &+ \frac{1}{AR(c^*(t))}\kappa_{v^*}(t)^\top\,dw(t). \end{aligned} \tag{2}$$

Hence

$$g^*(t) = \frac{1}{RR(c^*(t))}(r(t) + v_0^*(t) - \theta) + \frac{1}{2}\frac{RP(c^*(t))}{(RR(c^*(t)))^2}||\kappa_{v^*}(t)||^2. \tag{3}$$

The condition $W(t) \geqslant -\underline{W}$ implies that $v_0^*(t) \geqslant 0$. The growth rate for the Ramsey model can be obtained from the previous relationship by setting $v_0^* = 0$ and $\kappa_{v^*} = 0$ [16]. Since $u''' > 0$, we have $RP > 0$ and the desired conclusion follows. $\square$

As highlighted by Kimball [9] and Levhari [11] for a two-period framework, when the marginal utility function is convex ($u''' > 0$), agents have a *prudent* attitude and aim at preparing and forearming themselves in the face of uncertainty. This is the precautionary savings motive quantified here by the term $\frac{1}{2}\frac{RP(c^*(t))}{(RR(c^*(t)))^2}||\kappa_{v^*}(t)||^2$ in relationship (3). Prudence enhances precautionary savings whereas risk aversion reduces it. This result is to be related with the Drèze–Modigliani "substitution effect" [8]: in a two period model, "the reduction in first period consumption is larger than what one would expect by looking at the reduction in utility caused by income risk when preferences for second period consumption display decreasing absolute risk aversion" [9, p. 65]. In the framework presented here, it can be shown that if $u'''' < 0$ or RP is a decreasing (increasing) function and $RP(c) < RR(c)$ ($RP(c) > RR(c)$), then precautionary savings decreases (increases) with the level of consumption c. Agents can choose to reduce current consumption to save more to hedge against uncertainty. Higher savings lead to more growth. Using numerical methods, Zeldes [17] shows that in presence of transitory income, consumption displays an excess sensitivity, g^* is too high and c^* is too low.

3. Application

Assumption (A1). $u(c,t) = e^{-\theta t} \ln c$ and a, p, r, σ are *stochastic processes,* or u can be *any* utility function but a, p, r, σ are *deterministic functions of time* only.

3.1. Preliminary results

Proposition 2. *When the production function is homogenous of degree one, the set of constraints is a cone and preferences satisfy* (A1), *the dual program* (*DP*) *is fully characterized by the following relationships*

$$E\left[\int_0^T \tilde{u}_c(\psi^* \pi_{v^*}(t), t)\pi_{v^*}(t)\, dt\right] + W_0 = 0, \tag{i}$$

$$(v_0^*, v^*) = \arg \min_{(v_0,v)\in\mathcal{N}} \frac{1}{2}||\kappa_v||^2 + v_0. \tag{ii}$$

Proof. Condition (i) comes straightforwardly from the fact that $\forall (v_0, v) \in \mathcal{N}$, $e(v_0, v) = 0$ and the minimization with respect to the parameter ψ.

Logarithmic preferences. $\tilde{u}(y,t) = -e^{-\theta t}(1 + \theta t + \ln y)$ and we have to solve

$$\min_{(\psi,(v_0,v))\in\mathbb{R}_{++}\times\mathcal{N}} E\left[\int_0^T e^{-\theta t}\left(\int_0^t \left(\frac{||\kappa_v(u)||^2}{2} + v_0(u) + r(u)\right) du\right) dt\right] - \frac{1 - e^{-\theta T}}{\theta} \ln \psi + \psi W_0.$$

The solution is given by minimizing with respect to ψ and by minimizing *pointwise* with respect to (v_0, v) the functional $(v_0, v) \mapsto \frac{||\kappa_v||^2}{2} + v_0$.

Deterministic coefficients. Given ψ the minimization problem is

$$\min_{(v_0,v)\in\mathcal{N}} \quad E\left[\int_0^T \tilde{u}(\psi\pi_v(t), t)\, dt\right]$$

$$\text{s.t.} \qquad d\pi_v(t) = \pi_v(t)[-(r(t) + v_0(t))\, dt + \kappa_v(t)^\top\, dw(t)].$$

Define $J(\pi_v, t) = \min_{(v_0,v)\in\mathcal{N}} E[\int_t^T \tilde{u}(\psi\pi_v(s), s)\, ds \mid \mathcal{F}_t]$. Then J satisfies the Hamilton Jacobi Bellman (HJB) equation:

$$0 = J_t(\pi_v, t) + \min_{(v_0,v)\in\mathcal{N}} \tilde{u}(\psi\pi_v, t) - J_\pi(\pi_v, t)\pi_v(r + v_0) + \frac{1}{2} J_{\pi\pi}(\pi_v, t)\pi_v^2 ||\kappa_v||^2.$$

Because $\tilde{u}$ is strictly convex and decreasing, J is strictly convex and decreasing so $J_\pi < 0$ and $J_{\pi\pi} > 0$. Hence, the minimum is achieved exactly when v_0 and $v \mapsto \frac{||\kappa_v||^2}{2}$ are minimum. Note that v^* is a *deterministic* process. □

For (v_0, v) given, a complete characterization of the optimal policies is provided in Appendix A when preferences satisfy Assumption (A1).

Investment constraints and consumption growth rate. The more constraints imposed on capital inputs, the *smaller* is the set Q and consequently the *larger* is the effective domain $\mathcal{N}$. Therefore, the functional $(v_0, v) \mapsto \frac{\|\kappa_v\|^2}{2} + v_0$ can achieve a lower value so does g^*. *Thus, the more restrictions on capital inputs, the smaller precautionary savings are and consequently the lower the consumption growth rate.*

3.2. CRRA preferences and Cobb–Douglas technology

Using relationship (3) when $u(c,t) = \frac{c^{1-b}-1}{1-b} e^{-\theta t} (b>0)$ leads to

$$g^*(t) = \frac{1}{b}\left[r(t) - \theta + \frac{(1+b)}{2b} \|\kappa_{v^*}(t)\|^2 \right].$$

In the sequel, we assume that the production sector uses a Cobb–Douglas technology with two inputs $y = F(a', k_1, k_2) = a' k_1^{\alpha} k_2^{1-\alpha}$. Alternatively, $y = F(a, K_1, K_2) = a K_1^{\alpha} K_2^{1-\alpha}$, with $a = \frac{a'}{p_{k_1}^{\alpha} p_{k_2}^{1-\alpha}}$ being an aggregator for both real and monetary shocks. As proved in Appendix B, for $Q = \{(K_1, K_2, X) \in \mathbb{R}^3, K_1 \geqslant 0, K_2 \geqslant 0\}$, the effective domain is $\mathcal{N} = \{(v_0, v_1, v_2) \in \mathbb{R}^3 : v_0 = 0,\ v_1 > 0,\ v_2 > 0 : 1 \leqslant (\frac{v_1}{\alpha a})^{\frac{1}{1-\alpha}} (\frac{v_2}{(1-\alpha)a})^{\frac{1}{\alpha}}\}$. To keep things simple, we assume that $\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ is a diagonal matrix.[3] The dual program becomes

$$\min_{1 \leqslant (\frac{v_1}{\alpha a})^{\frac{1}{1-\alpha}} (\frac{v_2}{(1-\alpha)a})^{\frac{1}{\alpha}}} \frac{1}{2\sigma_1^2}(v_1 - r)^2 + \frac{1}{2\sigma_2^2}(v_2 - r)^2.$$

The solution of this program is given by Proposition 3.

Proposition 3. *When the risky technology is not productive enough*, $\alpha^{\alpha}(1-\alpha)^{1-\alpha} a < r$, *all the resources are allocated to the risk-free technology. The minimum is achieved for* $v_1^* = v_2^* = r$, $\frac{X}{W} = 1$ *and* $g^* = \frac{r-\theta}{b}$. *Alternatively, when* $r \leqslant \alpha^{\alpha}(1-\alpha)^{1-\alpha} a$, *the optimal solution of the Dual Problem is*

$$v_1^* = \frac{r + \sqrt{r^2 + 4\frac{\lambda \sigma_1^2}{1-\alpha}}}{2} \quad \textit{and} \quad v_2^* = \frac{r + \sqrt{r^2 + 4\frac{\lambda \sigma_2^2}{\alpha}}}{2},$$

where λ *is implicitly defined by*: $\frac{1}{1-\alpha} \ln \frac{r + \sqrt{r^2 + 4\frac{\lambda\sigma_1^2}{1-\alpha}}}{2\alpha a} + \frac{1}{\alpha} \ln \frac{r + \sqrt{r^2 + 4\frac{\lambda\sigma_2^2}{\alpha}}}{2(1-\alpha)a} = 0$ *and*

$$\frac{K_i^*}{W} = \frac{v_i^* - r}{b\sigma_i^2}, \ i = 1, 2.$$

[3] This corresponds to the case where the shocks affecting the two types of capital have a zero correlation.

$$g^* = \frac{1}{b}\left[r - \theta + \frac{(1+b)}{2b}\left[\frac{1}{4\sigma_1^2}\left(\sqrt{r^2 + 4\frac{\lambda\sigma_1^2}{1-\alpha}} - r\right)^2 + \frac{1}{4\sigma_2^2}\left(\sqrt{r^2 + 4\frac{\lambda\sigma_2^2}{\alpha}} - r\right)^2\right]\right].$$

Proof. See Appendix B. □

3.2.1. Comparative statics
Portfolio selection

Proposition 4. *When* $r < \alpha^\alpha (1-\alpha)^{1-\alpha} a$, *increasing* σ_1 (*respectively* σ_2) *keeping* σ_2 (*respectively* σ_1) *constant lowers the proportion of the wealth invested in the risky asset* 1 (*respectively risky asset* 2).

Proof. See Appendix B. □

This result corroborates Levhari and Srinivasan's conjecture [10, p. 163] that "when the variance of an asset is increased keeping its mean constant, the optimal proportion invested in this asset goes down." In addition, as shown in Appendix B, $\frac{\partial v_j^*}{\partial \sigma_i} < 0$, $i \neq j$, which implies that $\frac{\partial(\frac{K_j^*}{W})}{\partial \sigma_i} < 0$. The proportion of wealth invested in the risky asset j shrinks as the variance of the risky asset i rises because the marginal return of asset j decreases when the amount invested in asset i decreases. Consequently, the proportion of wealth invested into the risk-free asset must increase.

Uncertainty, risk aversion, consumption growth rate and savings rate

Proposition 5. *When* $r < \alpha^\alpha (1-\alpha)^{1-\alpha} a$, *uncertainty* (σ_1, σ_2) *has a negative impact on the consumption growth rate.*

Proof. Recall that $\mathcal{N}$ is independent of (σ_1, σ_2) so using the *envelope theorem* we can write $\frac{dg^*}{d\sigma_i} = \frac{\partial g^*}{\partial \sigma_i} = \frac{(1+b)}{2b^2}\frac{\partial ||\kappa_{v^*}||^2}{\partial \sigma_i} = -\frac{(1+b)}{2b^2}\frac{1}{\sigma_i^3}(v_1^* - r)^2 < 0$, $i = 1, 2$. □

Ramey and Ramey [15] found that countries with higher volatility have a lower growth. Mendoza [12] also concluded in a negative relationship between growth and the terms-of-trade volatility in the case of small risk aversion preferences. In addition, as shown in Appendix A, the ratio consumption over wealth is given by

$$\frac{c^*(t)}{W(t)} = \left(\int_t^T \exp\int_t^s \left[-\frac{1}{b}\left(\theta + (b-1)\left(r(u) + \frac{b-1}{2b}||\kappa_{v^*}(u)||^2\right)\right) du\right] ds\right)^{-1}.$$

Given what precedes, if $r \leqslant \alpha^{\alpha}(1-\alpha)^{1-\alpha}a$, the ratio consumption–wealth ratio decreases (increases) when uncertainty parameters σ_i, $i = 1, 2$ rise exactly when $\frac{1}{b} < 1 (\frac{1}{b} > 1)$, reflecting consumption smoothing motives since b equals to the inverse of the intertemporal elasticity of substitution. Consequently, the more uncertainty, the higher (lower) the savings rate exactly when $\frac{1}{b} < 1 (\frac{1}{b} > 1)$. Finally, it is easy to check that, as in the Ramsey model, the consumption growth rate decreases with b if $r > \theta$ [16].

Interest rate and consumption growth rate. Zeldes [17] points out that the aggregate consumption growth rate can remain positive for long period of time despite a very low real interest rate, in contradiction with the predictions of the Ramsey model [16]. Proposition 6 reconciliates the empirical findings of Zeldes.

Proposition 6. *Introducing uncertainty leads to a U-shaped relationship between the real interest rate and consumption growth rate. The latter admits a minimum at r^* that decreases with risk aversion b and the risk magnitude σ_1 and σ_2. At r^*, the fraction of wealth invested in the risk-free asset only depends on b and it is increasing in b.*

Proof. When $r \leqslant \alpha^{\alpha}(1-\alpha)^{1-\alpha}a$ then $g^* = \frac{1}{b}[r - \theta + \frac{(1+b)}{2b}||\kappa_{v^*}||^2]$ and $\mathcal{N}$ is independent of r. Using the *envelope theorem* leads to $\frac{dg^*}{dr} = \frac{\partial g^*}{\partial r} = \frac{1}{b}[1 - \frac{(1+b)}{b}[\frac{1}{\sigma_1^2}(v_1^* - r) + \frac{1}{\sigma_2^2}(v_2^* - r)]]$. In Appendix C, we show that $r \mapsto \frac{dg^*}{dr}$ admits a *unique* root denoted r^*, decreasing in b, σ_1 and σ_2 and when $r = r^*$, then $\frac{X}{W} = \frac{b}{1+b}$. $\square$

At $r = r^*$ the fraction of wealth invested in the risk-free asset is increasing in b. A myopic agent $(b = 1)$ *equally* splits her wealth between the risk-free asset and the risky technology. When the coefficients b, σ_1 and σ_2 are high, the individual prefers to rely *mainly* on the risk-free asset provided that $r > r^*$: The consumption growth rate is increasing with the interest rate as in the standard Ramsey model [16].

Autarchy. We consider a closed economy that uses a unique risky technology for production. This case can be seen as a special case of what precedes by adjusting the interest rate r in such a way that no resource is devoted to the risk-free technology. Setting $X \equiv 0$ yields $b = \frac{1}{\sigma_1^2}(v_1^*(r) - r) + \frac{1}{\sigma_2^2}(v_2^*(r) - r)$. The existence and the uniqueness of such an interest rate is easy to show as $r \mapsto \frac{1}{\sigma_1^2}(v_1^*(r) - r) + \frac{1}{\sigma_2^2}(v_2^*(r) - r)$ is continuous and strictly decreasing from $+\infty$ to 0. For the sake of simplicity, we present the special tractable case where $\frac{\sigma_1^2}{1-\alpha} = \frac{\sigma_2^2}{\alpha}$. Then, we have $v_1^* = v_2^* = \alpha^{\alpha}(1-\alpha)^{1-\alpha}a$ so $r = \alpha^{\alpha}(1-\alpha)^{1-\alpha}a - \alpha\sigma_1^2 b$ and

$$g^* = \frac{1}{b}\left[\alpha^{\alpha}(1-\alpha)^{1-\alpha}a - \theta + \alpha\frac{b(b-1)}{2}\sigma_1^2\right].$$

As a function of b, the growth rate now has a U shape, decreasing when b is small and increasing when b is large. When preferences admit an expected utility representation, it is not possible to distinguish the intertemporal substitution of consumption from risk aversion. A slight adaptation of the example worked out by

Dumas et al. [8] who uses Kreps–Porteus' recursive preferences, suggests that the consumption growth rate is given by

$$g^* = s\left[\alpha^\alpha(1-\alpha)^{1-\alpha}a - \theta + \frac{1-s}{s}\frac{\alpha b\sigma_1^2}{2}\right],$$

where s is the IES and b is the coefficient of risk aversion. Then, it becomes clear that the effect of risk aversion is to magnify uncertainty. The cut off point $s=1$ (myopic agent) governs the impact of uncertainty on g^*. The term $(1-s)\frac{\alpha b\sigma_1^2}{2}$ encapsulates precautionary savings motive. Then, it should come as a no surprise that when agents' IES is high enough $(s>1)$, more uncertainty lowers the consumption growth rate (lower precautionary savings). The opposite applies when $(s<1)$.

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Appendix A

For any *deterministic* process (v_0, v), the optimal condition, $c^*(t)^{-b}e^{-\theta t} = \psi^*\pi_v(t)$ implies that $W(t) = \pi_v(t)^{-1}E(\int_t^T \psi^{-\frac{1}{b}}\pi_v(s)^{\frac{b-1}{b}}e^{-\frac{\theta}{b}s}\,ds \mid \mathscr{F}_t)$. The optimal condition on v_0 is: $v_0^* = 0$. Moreover, for $t<s$, $\pi_v(s) = \pi_v(t)\exp(\int_t^s -(r(u) + \frac{1}{2}||\kappa_v(u)||^2)\,du + \kappa_v(u)^\top\,dw(u))$. Since κ_v is a deterministic process, it follows that

$$W(t) = \pi_v(t)^{-\frac{1}{b}}\psi^{-\frac{1}{b}}\int_t^T \exp\int_t^s \times\left[-\frac{1}{b}\left(\theta + (b-1)\left(r(u) + \frac{b-1}{2b}||\kappa_v(u)||^2\right)\right)du\right]ds.$$

Thus, the Lagrange multiplier ψ^* and the consumption c^* are given by

$$\psi^* = W_0^{-b}\left(\int_0^T \exp\int_0^s\left[-\frac{1}{b}\left(\theta + (b-1)\left(r(u) + \frac{b-1}{2b}||\kappa_v(u)||^2\right)\right)du\right]ds\right)^b,$$

$$\frac{c^*(t)}{W(t)} = \left(\int_t^T \exp\int_t^s\left[-\frac{1}{b}\left(\theta + (b-1)\left(r(u) + \frac{b-1}{2b}||\kappa_v(u)||^2\right)\right)du\right]ds\right)^{-1}.$$

Finally, by Ito's lemma, $dW(t) = \mu_W(t)\,dt - \frac{W(t)}{b}\kappa_v(t)^\top\,dw(t)$ for some process μ_W. Identifying the coefficients from relationship (1) yields $K_i(t) = \frac{v_i(t)-r(t)}{b\sigma_i^2(t)}W(t)$, $i = 1, 2$. When $b = 1$, direct calculations show that $W(t) = W_0\frac{e^{-\theta t}-e^{-\theta T}}{1-e^{-\theta T}}\pi_v(t)^{-1}$ and $c(t) = \frac{\theta}{1-e^{-\theta(T-t)}}W(t)$.

Appendix B

The function $(X, K_1, K_2) \mapsto aK_1^\alpha K_2^{1-\alpha} - v_0X - v_1K_1 - v_2K_2$ is homogenous of degree one so its supremium is finite exactly when it is non-positive. This yields $v_0 = 0$ and $v_i \geqslant 0$, $i = 1, 2$. Given $K_2 \geqslant 0$, the first order condition with respect to K_1 is: $K_1 = (\frac{\alpha a}{v_1})^{\frac{1}{1-\alpha}}K_2$. Plugging back into the objective function provides the desired conclusion.

Proof of Proposition 3. When $\alpha^\alpha(1-\alpha)^{1-\alpha}a \leqslant r$, the obvious solution is $(v_1^*, v_2^*) = (r, r)$ and $\kappa_{v^*} = 0$ as $1 \leqslant (\frac{r}{\alpha a})^{\frac{1}{1-\alpha}}(\frac{r}{(1-\alpha)a})^{\frac{1}{\alpha}}$. Thus $K_i(t) = 0$, $i = 1, 2$ and $X(t) = W(t)$. When $r < \alpha^\alpha(1-\alpha)^{1-\alpha}a$, the minimization program can be written as

$$\min_{A \leqslant \frac{1}{1-\alpha}\ln v_1 + \frac{1}{\alpha}\ln v_2} \frac{1}{2\sigma_1^2}(v_1 - r)^2 + \frac{1}{2\sigma_2^2}(v_2 - r)^2,$$

where $A = \frac{1}{1-\alpha}\ln(\alpha a) + \frac{1}{\alpha}\ln((1-\alpha)a)$. Let λ be the Lagrange multiplier associated with this program. The first-order conditions are $\frac{1}{\sigma_1^2}(v_1^* - r) = \frac{\lambda}{1-\alpha}\frac{1}{v_1^*}$, $\frac{1}{\sigma_2^2}(v_2^* - r) = \frac{\lambda}{\alpha}\frac{1}{v_2^*}$. Given λ, $v_1^* = \frac{r+\sqrt{r^2+\frac{4\sigma_1^2\lambda}{1-\alpha}}}{2}$ and $v_2^* = \frac{r+\sqrt{r^2+\frac{4\sigma_2^2\lambda}{\alpha}}}{2}$ and λ is uniquely determined by the relationship:

$$0 = \frac{1}{1-\alpha}\ln\frac{r + \sqrt{r^2 + \frac{4\lambda\sigma_1^2}{1-\alpha}}}{2\alpha a} + \frac{1}{\alpha}\ln\frac{r + \sqrt{r^2 + \frac{4\lambda\sigma_2^2}{\alpha}}}{2(1-\alpha)a}. \tag{A.1}$$

The rest of the proof follows easily. □

Proof of Proposition 4. Since $\mathcal{N}$ is independent of (σ_1, σ_2) using the *envelope theorem* implies that $\frac{1}{\sigma_1^2}(v_1^* - r)\frac{\partial v_1^*}{\partial \sigma_1} + \frac{1}{\sigma_2^2}(v_2^* - r)\frac{\partial v_2^*}{\partial \sigma_1} = 0$. Since $v_i^* - r$, $i = 1, 2$ are positive, $\frac{\partial v_1^*}{\partial \sigma_1}$ and $\frac{\partial v_2^*}{\partial \sigma_1}$ must have opposite signs. Recall that $\frac{1-\alpha}{\sigma_1^2}(v_1^* - r)v_1^* = \frac{\alpha}{\sigma_2^2}(v_2^* - r)v_2^*$. Totally differentiating with respect to σ_1 yields

$$-\frac{2(1-\alpha)}{\sigma_1^3}(v_1^* - r)v_1^* = -\frac{1-\alpha}{\sigma_1^2}(2v_1^* - r)\frac{\partial v_1^*}{\partial \sigma_1} + \frac{1-\alpha}{\sigma_1^2}(2v_2^* - r)\frac{\partial v_2^*}{\partial \sigma_1}. \tag{A.2}$$

The LHS of relationship (A.2) is negative. Since $2v_i^* - r$, $i = 1, 2$ are positive and $\frac{\partial v_1^*}{\partial \sigma_1}$ and $\frac{\partial v_2^*}{\partial \sigma_1}$ must have opposite signs, we have $\frac{\partial v_1^*}{\partial \sigma_1} > 0$. Then from $\frac{K_1}{W} = \frac{v_1^* - r}{b\sigma_1^2} = \frac{\alpha}{(1-\alpha)\sigma_2^2}(v_2^* - r)\frac{v_2^*}{v_1^*}$, it follows that $\frac{\partial \ln(\frac{K_1}{W})}{\partial \sigma_1} = (\frac{1}{v_2^* - r} + \frac{1}{v_2^*})\frac{\partial v_2^*}{\partial \sigma_1} - \frac{1}{v_1^*}\frac{\partial v_1^*}{\partial \sigma_1}$. Hence, given what precedes, $\frac{\partial \ln(\frac{K_1}{W})}{\partial \sigma_1} < 0$. □

Appendix C

Preliminary results. Applying the *implicit function theorem* to relationship (A.1) shows that $r \mapsto \lambda'(r)$ is well defined on $I = (-\infty, \alpha^\alpha(1-\alpha)^{1-\alpha}a]$ and satisfies $\frac{1 + \frac{r + 4\frac{\lambda'(r)\sigma_1^2}{1-\alpha}}{\sqrt{r^2 + 4\frac{\lambda(r)\sigma_1^2}{1-\alpha}}}}{r + \sqrt{r^2 + 4\frac{\lambda(r)\sigma_1^2}{1-\alpha}}} + \frac{1 + \frac{r + 4\frac{\lambda'(r)\sigma_2^2}{\alpha}}{\sqrt{r^2 + 4\frac{\lambda(r)\sigma_2^2}{\alpha}}}}{r + \sqrt{r^2 + 4\frac{\lambda(r)\sigma_2^2}{\alpha}}} = 0$. It is easy to check that $\forall r \in I$, $\lambda'(r) < 0$. By continuity, $\lambda(\alpha^\alpha(1-\alpha)^{1-\alpha}a) = 0$. Moreover, since λ is monotonic, it admits a limit in $-\infty$. Now, assume that $\lim_{-\infty} \lambda = l \in \mathbb{R}_{++}$. For $\gamma > 0$, we have $r + \sqrt{r^2 + 4\lambda(r)\gamma} \underset{-\infty}{=} -\frac{2\gamma\lambda(r)}{r} + o(1/r)$, so $\lim_{-\infty} v_1^*(r) = \lim_{-\infty} v_2^*(r) = 0$, which violates relationship (A.1). Therefore $\lim_{-\infty} \lambda = +\infty$. Then, we establish that $r \mapsto v_i^*(r) - r$, $i = 1, 2$ are strictly decreasing functions on I. Define $\varphi(r) = 2(v_1^*(r) - r)$. Note that $\varphi(r) > 0$ and satisfies $0 = \frac{1}{1-\alpha}\ln\frac{2r + \varphi(r)}{2\alpha a} + \frac{1}{\alpha}\ln\frac{r + \sqrt{(1-u)r^2 + u(\varphi(r) + r)^2}}{2(1-\alpha)a}$, where $u = \frac{(1-\alpha)\sigma_2^2}{\alpha\sigma_1^2} > 0$. Totally differentiating with respect to r yields $\frac{1}{1-\alpha}\frac{2 + \varphi'(r)}{2r + \varphi(r)} + \frac{1}{\alpha}\frac{1 + \frac{u(1 + \varphi'(r))(\varphi(r) + r) + 1}{\sqrt{(1-u)r^2 + u(\varphi(r) + r)^2}}}{r + \sqrt{(1-u)r^2 + u(\varphi(r) + r)^2}} = 0$. It is then easy to check that $\varphi'(r) < 0$. Same proof for $r \mapsto v_2^*(r) - r$. □

Proof of Proposition 5. Given what precedes, $r \mapsto \frac{dg^*}{dr} = \frac{1}{b}[1 - \frac{(1+b)}{b}[\frac{1}{\sigma_1^2}(v_1^* - r) + \frac{1}{\sigma_2^2}(v_2^* - r)]]$ is continuous and strictly increasing on I, $\lim_{-\infty} \frac{dg^*}{dr} = -\infty$ and $\frac{dg^*}{dr}(\alpha^\alpha(1-\alpha)^{1-\alpha}a) = \frac{1}{b}$. Thus, g^* has a *unique* minimum r^*, characterized by $\frac{b}{1+b} = \frac{1}{\sigma_1^2}(v_1^*(r^*) - r^*) + \frac{1}{\sigma_2^2}(v_2^*(r^*) - r^*)$. Differentiating both sides with respect to b yields $\frac{1}{(1+b)^2} = \frac{d}{dr}[\frac{1}{\sigma_1^2}(v_1^*(r^*) - r^*) + \frac{1}{\sigma_2^2}(v_2^*(r^*) - r^*)]\frac{\partial r^*}{\partial b}$. Since $G(r) \equiv \frac{d}{dr}[\frac{1}{\sigma_1^2}(v_1^*(r^*) - r^*) + \frac{1}{\sigma_2^2}(v_2^*(r^*) - r^*)] < 0$, we have $\frac{\partial r^*}{\partial b} < 0$. Using similar arguments leads to $\frac{\partial r^*}{\partial \sigma_1} < 0$ and $\frac{\partial r^*}{\partial \sigma_2} < 0$. □

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