Asset Selection and Under-Diversification with Financial Constraints and Income: Implications for Household Portfolio Studies

Hervé Roche and Stathis Tompaidis*
Centro de Investigación Económica
Instituto Tecnológico Autónomo de México
Av. Camino a Santa Teresa No 930
Col. Héroes de Padierna
10700 México, D.F.
E-mail: hroche@itam.mx

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Abstract

We offer a rational explanation the observed under-diversification of household portfolio in a complete market, partial equilibrium setting with an investor with CRRA preferences, whose investment opportunity set includes both a riskless asset and multiple risky assets, and who receives an income stream. We show that when the investor faces a margin requirement based on his current wealth, he shifts his portfolio towards under-diversified portfolios with fewer assets that offer higher expected returns. We identify the ratio of financial wealth to financial wealth augmented by discounted lifetime labor income as the variable that governs the investor’s behavior.

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*Roche is with the Centro de Investigación Económica, ITAM, hroche@itam.mx. Tompaidis is with the McCombs School of Business, University of Texas at Austin, Department of Information, Risk and Operations Management, stathis.Tompaidis@mccombs.utexas.edu. We would like to thank seminar participants at the University of Texas at Austin for helpful comments and suggestions.
The problem of individual optimal asset allocation and consumption selection in a partial equilibrium setting has been studied extensively in the literature, starting with the work of Merton (1971). The standard framework suggests that risk aversion determines the mix of risky and riskless assets in an individual’s portfolio while within the risky asset portfolio all investors hold risky assets in the same proportions. The prescription, often called mutual fund separation theorem, has partly been the reason for the explosive growth in the size of mutual funds that track the market portfolio over the last 30 years.\footnote{See Cass and Stiglitz (1970).}

In this paper, we extend the literature by considering an investor that receives income and faces financial constraints, in the form of margin requirements for allocations in risky assets.\footnote{The combination of margin requirements and income has previously been considered in the literature in Cuoco (1997), where existence results were provided for the optimal consumption and allocation strategies.} The margin requirements can be satisfied only out of the current wealth of the investor, effectively rendering future earnings non-tradable. This restriction is due to adverse selection and moral hazard problems, as well as the inalienability of human capital. We are able to characterize the optimal investment and consumption strategies and uncover a few surprising facts. Regarding asset allocation, we find that when the margin constraint is binding mutual fund separation no longer holds. Instead, the investor chooses to stay away from a diversified portfolio and shifts his portfolio towards assets with high expected returns, with little concern for their risk. This risk-taking behavior is an attempt to compensate for the margin constraints he faces, which limits the overall exposure of the investor to risky assets. Intuitively, this behavior can be understood by considering the case of an unconstrained investor: such an investor follows an investment strategy that can be characterized in terms of effective wealth, which adds discounted future earnings to current wealth and allocates a constant percentage of the effective wealth into a mutual fund of the risky assets with constant composition. Due to the financial constraints, such a strategy may be unavailable to the constrained investor when current wealth is only a small fraction of the effective wealth forcing the constrained investor to reduce his allocation in risky assets. The investor is then left to balance his diversification motive, with his motive for higher returns. These conflicting motives lead him to engage in asset substitution, which, in the case of investors that are severely constrained, leads to selecting assets primarily based on their expected returns, resulting in deviations from the diversified portfolio. This behavior can also be understood in terms of an effective coefficient of risk aversion, which decreases as the margin constraint...
becomes more binding. 3

Our findings can be translated into empirically testable hypotheses and compared to findings in the empirical literature. The clearest predictions occur for the case of young investors, since they are likely to find the margin constraint binding. Undiversified portfolios, low savings rates and limited stock market participation are in line with our model.

Kelly (1995) studies the 1983 Survey of Consumer Finances and finds that diversification increases with portfolio size, age, and wealth. Polkovichencho (2005) uses the 1983, 1989, 1992, 1995, 1998 and 2001 Survey of Consumer Finances and confirms that wealthier households hold more diversified portfolios, even though not all wealthy households are well diversified. He argues that investors are aware of the higher risk associated with undiversified portfolios and proposes preferences with rank dependency as a potential explanation. Our results show that standard utility-based preferences in the presence of financial constraints can result in similar allocation patterns.

Ivkovic, Sialm, and Weisbenner (2004) use data from trades and monthly portfolio positions of retail investors at a large U.S. discount brokerage house for the 1991-1996 period. They show that the number of stocks in the portfolio increases with the size of the account balance, and that concentrated portfolios have higher levels of risk and return and lower Sharpe ratios than diversified portfolios. Goetzmann and Kumar (2005) study the same dataset and find that diversification increases with age and income, while households with only a retirement account hold less diversified portfolios than households with additional non-retirement investment accounts. They examine several potential explanations for the lack of diversification: small portfolio size and transaction costs; search and learning costs; investor demographics and financial sophistication; layered portfolio structure; preference to higher order moments; and behavioral biases such as illusion of control and investor over-confidence, local bias and trend-following behavior. Kumar (2005), using the same dataset, finds that young investors have a strong preference for riskier stocks, and argues that the young are more likely to be heavy lottery players, and this is reflected in their selection of stocks. Our results offer a potential and complementary explanation to these empirical findings, based on the rational behavior of investors facing binding financial constraints.

3In addition to changes in the asset allocation, the constraint also induces changes in the consumption behavior. We show that, in a model that considers an infinitely lived investor that receives an uninterrupted income stream, when the wealth of the investor is zero the investor’s consumption rate equals his income rate, preventing wealth accumulation. While this result depends on the assumptions of infinite horizon and income, it does suggest that investors with relatively little wealth and relatively long life and large income, have little incentive to save.
In a recent paper, Calvet, Campbell, and Sodini (2006) study a dataset of the portfolios of the entire Swedish population and propose several measures to quantify the under-diversification of household portfolios. They show that increasing age, wealth and financial sophistication increase diversification, but also lead to investors taking more aggressive positions. We use the measures they develop to quantify the magnitude of under-diversification that can result from facing a binding financial constraint.

Literature related to the techniques used in our paper includes the papers by Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989) who introduce martingale techniques that make it easier to deal with constraints on the investment strategies. We use the same techniques in our paper. Models with constraints on the portfolio policies are studied by Karatzas, Lehoczky, Shreve, and Xu (1991), Cvitanic and Karatzas (1992), Cvitanic and Karatzas (1993), He and Pearson (1991) and Shreve and Xu (1992). Cuoco (1997) is able to demonstrate existence of optimal strategies for the case of an investor that faces margin constraints and receives income but does not provide a characterization of the strategies. Cuoco and Liu (2000) discuss the case of an investor facing margin requirements but does not receive income, and provide a characterization of his optimal investment strategy. We extend their work, and illustrate that their results imply that asset substitution may occur even for investors that do not receive income, as long as the margin constraint is binding. He and Pages (1993), El Karoui and Jeanblanc-Piqué (1998) and Duffie, Fleming, Soner, and Zariphopoulou (1997) study the optimal asset selection problem of an investor who receives income and who is constrained to maintain non-negative levels of current wealth, but do not address margin requirements.4 Detemple and Serrat (2003) solve the consumption-portfolio problem within a general equilibrium framework in presence of liquidity constraints as some investors are restricted from borrowing against their future labor income. In particular, they find that constrained agents aim at deferring consumption at the early stages of their life in order to build up liquid wealth. Finally, liquidity constraints lower the equilibrium interest rate but leave unchanged the Sharpe ratio and therefore fail to explain the equity premium puzzle.

The remainder of the paper is organized as follows: in Section 1 we present a two period model in order to illustrate the intuition of our results in a simple setting. Section 2 presents a continuous time model, and demonstrates that the intuition of the simple model carries through. Section 3 describes quantitative measures of diversification loss in numerical simulations. Section 4 summarizes

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4 In contrast to our work, they show that when current wealth is zero, the investor consumes only a fraction of his income, indicating that, compared to the constraint of non-negative wealth, margin requirements are much more stringent.
and concludes the paper. The proofs are contained in Appendix A.

1 A Two Period Model

To build intuition regarding the interpretation of our results, we consider a two period discrete time model. The economy consists of one riskless and two risky assets. The investor has logarithmic preferences and does not discount future consumption. He starts in the first period with initial wealth \( W > 0 \), and receives, in the second period, income \( Y \). In the first period the investor consumes \( c_0 \), and chooses allocations in the risky assets \( z_1, z_2 \), and in the riskless asset \( x \), where \( c_0, z_1, z_2, x \) are measured in dollars. In addition, there are margin constraints in the allocations of the risky assets

\[
z_1 + z_2 \leq W - c_0, z_1 \geq 0, z_2 \geq 0,
\]

i.e., the investor can only hold the risky assets long, and cannot borrow against the value of the risky assets. This constraint also implies that \( c_0 \leq W \), i.e., the investor cannot borrow to finance consumption. In the second period the investor consumes his total wealth, made up by the income \( Y \), and the value of his financial portfolio. Setting the interest rate on the riskless asset to zero, normalizing prices of the risky assets in the first period to 1, and denoting prices in the second period by \( S_1, S_2 \), the investor’s wealth in the second period is \( z_1 S_1 + z_2 S_2 + x + Y \). For simplicity, we assume that the prices of the risky assets can take only three values. We assume that the returns of the assets are 5% for asset 1 and 3% for asset 2, and the standard deviation of returns 60% and 30% respectively. We also assume that the returns of the assets are independent. A set of prices that satisfies these criteria is

\[
(S_1, S_2) = \begin{cases} 
(1.891, 0.975) \text{ in state 1} \\
(0.724, 1.422) \text{ in state 2} \\
(0.535, 0.693) \text{ in state 3,}
\end{cases}
\]

with the probability of each state equal to 1/3. The investor chooses consumption and asset allocation in the first period to maximize expected utility

\[
\max_{(c_0, x, z_1, z_2)} \log c_0 + E \log c_1 \\
\text{s.t.} \quad c_0 + z_1 + z_2 + x = W \\
\quad c_1 = x + z_1 S_1 + z_2 S_2 + Y \\
\quad z_1 \geq 0, z_2 \geq 0 \\
\quad z_1 + z_2 \leq W - c_0
\] (1)
The solution of the optimization problem (1) is presented in Figure 1. Details are presented in Appendix A. From the figure, we note that there are several regions where the investor’s behavior is qualitatively different. These regions are characterized in terms of the ratio of the investor’s current wealth $W$, over his effective wealth $W + Y$, which includes future wealth from income: (a) when current wealth is above 72% of the effective wealth, the margin constraint is not binding and the investor holds both risky assets, as well as the riskless asset in positive amounts; (b) when current wealth is between 72% and 52% of effective wealth, the investor no longer holds the riskless asset. In this region, as the ratio of wealth to effective wealth decreases, the investor shifts his portfolio towards the first asset. The intuition behind this substitution is that the investor is constrained from choosing a portfolio with enough exposure to the risky assets and trades the higher return of the first risky asset against the diversification benefit from holding both assets in the proportions chosen by the unconstrained investor; (c) when the current wealth to effective wealth ratio is between 52% and 49%, the investor’s portfolio consists of one risky asset only, while (d) for current wealth to effective wealth ratios below 49%, the investor consumes his entire wealth and does not invest in either the risky or the riskless assets.

Intuitively, we can summarize the investor’s behavior by considering the wealth to effective wealth ratio: for large values of the ratio, the investor follows the same investment strategy as an investor who does not face a margin constraint, while, when, for smaller values of the ratio the margin constraint binds, the investor is forced to reduce the size of his positions and shifts towards the asset with the higher expected return.

2 Continuous Time Model

While the two period example presented in Section 1, demonstrates the intuition behind the behavior of an investor that faces margin requirements as the ratio of his initial wealth to his income varies, it is not general enough to draw definite conclusions from - for example, it does not account for intertemporal hedging motives of the investor. In this section, we present a more general, continuous time, model and demonstrate that the intuition developed in Section 1 carries through.
2.1 The Economic Setting

We consider a continuous time economic setting, with an infinitely lived investor who derives utility from consumption and who is able to invest in a riskless bond and two, independent, risky assets that evolve according to geometric Brownian motion with constant coefficients.\(^5\) We will assume that the investor’s utility is of the CRRA type.

The Financial Market. Uncertainty is modeled by a probability space \((\Omega, \mathcal{F}, P)\) on which a two dimensional (standard) Brownian motion \(w = (w_1, w_2)\) is defined. A state of nature \(\omega\) is an element of \(\Omega\). \(\mathcal{F}\) denotes the tribe of subsets of \(\Omega\) that are events over which the probability measure \(P\) is assigned. In addition the filtration generated by the Brownian motions is denoted by \(\mathbb{F}\). In our setting, there are only two risky securities and a riskless bond available in the financial market. The value of the bond, \(B\) evolves according to

\[
dB_t = rB_t dt,
\]

where \(r\) is the constant interest rate. The two risky, non-dividend paying securities, with prices \(S_i\) follow geometric Brownian motion with constant coefficients

\[
dS_{it} = S_{it} (\mu_i dt + \sigma_i dw_{it}), \quad i = 1, 2,
\]

where \((dw_{1t}, dw_{2t})\) are the increments of two independent standard Wiener processes, \(\mu_i\) is the mean return of stock \(i\) and \(\sigma_i^2\) is its instantaneous variance.

Trading Strategies and Margin Requirements. We assume that consumption \(c_t\) and trading strategies \((x_t, z_t)\) are adapted processes satisfying the standard integrability conditions

\[
\int_0^\infty c_t^2 dt < \infty, \quad \int_0^\infty \|rx_t\| dt + \|z_t^T \mu\| dt + \|z_t^T \sigma\|^2 dt < \infty, \tag{2}
\]

where \(\mu^T = (\mu_1, \mu_2)\), \(\sigma = \text{diag}(\sigma_1, \sigma_2)\) and \(\|X\|\) is the the norm of \(X\), defined by \(\|X\|^2 = \text{Trace}(X^T X)\). \(x\) is the dollar amount invested in the riskless bond and \(z^T = (z_1, z_2)\), are the dollar amounts invested in the two risky assets.

To trade in risky assets, U.S. investors must hold sufficient wealth in a margin account. This wealth can be held in securities or cash. The Federal Reserve Board’s Regulation T sets the initial margin

\(^5\)We have chosen geometric Brownian motion for tractability reasons. The independence assumption is not critical, since independent assets can be constructed by imperfectly correlated ones, but simplifies the calculations and intuition.
requirement for stock positions undertaken through brokers. The current values for the initial margin
requirement are 50 percent for a long equity position, and 150 percent for a short equity position.\footnote{See Fortune (2000) as well as the Federal Reserve Board’s Regulation T for institutional details. In addition to our
discussion on initial margin requirements, there are also maintenance margin requirements, that correspond to the level
in the margin account at which collateral needs to be added to the account to avoid liquidation of the position. Including
a maintenance margin would make the problem path dependent and we do not consider it in this paper. The collateral
held in the margin account does not, in general, earn the risk free rate of interest, although large investors are able to
earn the \textit{general collateral rate}, see Geczy, Musto, Reed (2002). For the paper, we assume that the collateral held in the
margin account earns the risk-free interest rate $r$.}

For our model, we impose the following margin constraint on an investor that holds $z_i, i = 1, 2$
dollar amounts in the risky assets

$$\lambda^+(z_1^+ + z_2^+) + \lambda^-(z_1^- + z_2^-) \leq W,$$  \hspace{1cm} (3)

with $0 \leq \lambda^+ \leq 1, 0 \leq \lambda^-$.\footnote{For any real number $x$, we have $x = x^+ - x^-$, with $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.} The regulation T initial margin requirements correspond to $\lambda^+ = 0.5, \lambda^- = 1.5$. The constraint can be rewritten as

$$\sum_{i=1}^{2} \lambda_i z_i \leq W,$$

where $\lambda_k = \lambda^+$ if $z_k \geq 0$ and $\lambda_k = -\lambda^-$ if $z_k < 0$, for $k = 1, 2$. We note that the margin constraint is
more stringent than the constraint of non-negative wealth $W \geq 0$.

\textbf{Income.} We assume that the investor receives a non-negative income stream at a rate $Y(t)$, which
may be stochastic and is spanned by the risky assets in the economy.

$$dY_t = Y_t (mdt + \Sigma^\top dw_t)$$

where $m$ is the growth rate of income, and $\Sigma^\top = (\Sigma_1, \Sigma_2)$. All the coefficients are assumed to be constant.

\textbf{Preferences.} There is a single perishable good available for consumption, the numéraire. Preferences are represented by a time additive utility function

$$U(c) = E \left[ \int_0^\infty u(c_t)e^{-\theta t}dt \right],$$

where the time discount factor, $\theta$, is constant. The utility function $u$ is of the CRRA type, with risk
aversion coefficient $\gamma$:

$$u(c) = \begin{cases} 
\frac{c^{1-\gamma}}{1-\gamma} & , \quad \gamma \neq 1 \\
\ln c & , \quad \gamma = 1
\end{cases}$$
Optimization Problem. The investor’s problem is to maximize his discounted utility of consumption

$$F(W_t, Y_t) = \max_{(c \in C, (x, z) \in Q)} E_t \left[ \int_t^\infty u(c_s)e^{-\theta(s-t)}ds \right]$$

$$dW_s = \left( rW_s - c_s + Y_s + z^\top_s(\mu - r^T) \right)ds + \sigma z^\top_s dw_s$$

$$dY_s = Y_s \left( md_s + \Sigma^\top dw_s \right),$$

with $T^T = (1, 1)$, $Q$ the set of constraints imposed on the trading strategies $(x, z)$, $C$ the set of consumption plans that satisfy the integrability condition (2), and $W_t > 0, Y_t > 0$ the initial conditions for the investor’s wealth and income rate.

Transversality Condition. The transversality condition for this problem is given by

$$\lim_{T \to \infty} E_t \left[ F(W_{t+T}, Y_{t+T})e^{-\theta(T+t)} \right] = 0.$$

To ensure that the transversality condition is satisfied we will make certain assumptions on the coefficients of the stochastic processes later in the paper.

Properties of the Primal Value Function $F$. The value function $F$ has the following properties.

Proposition 1 $F$ is homogenous of degree $1 - \gamma$ in $(W, Y)$.

The homogeneity of $F$ allows us to rewrite the value function in terms of the ratio of wealth over income, as

$$F(W, Y) = Y^{1-\gamma}f\left(\frac{W}{Y}\right),$$

for some non-decreasing smooth function $f$. Below, we denote by $v$ the ratio of wealth over income.

Proposition 2 $F$ is non-decreasing in $W$ and $Y$ and jointly concave in $(W, Y)$

The proofs of the propositions can be found in Appendix A. Notice that the concavity of $F$ in $(W, Y)$ implies that $f$ is concave in $v$.

Dual Formulation. An alternative approach to the optimization problem (4) converts the (primal) dynamic problem into an equivalent (dual) static problem. The methodology we use relies on duality.
techniques developed by Cvitanic and Karatzas (1992) and Cuoco (1997). Let \( \pi \) be the state price density, defined by

\[
d\pi_t = \pi_t \left( -(r + a_t) dt - \left( \sigma^{-1} \left( b_t - a_t \mathbf{1} + \mu - r \mathbf{1} \right) \right)^\top dw_t \right),
\]

where \( a, b \) are adapted processes and \( \pi_0 = 1 \).

**Effective Domain.** For \( (a, b) \in \mathbb{R} \times \mathbb{R}^2 \), let \( e(a, b) \) be defined by

\[
e(a, b) = \sup_{(x,z) \in Q} - ax - \sum_{i=1}^{2} b_i z_i,
\]

where \( Q \) is a convex set of constraints imposed on the trading strategies. The effective domain \( \mathcal{N} \) is defined by

\[
\mathcal{N} = \{ (a, b) \in \mathbb{R} \times \mathbb{R}^2, e(a, b) < \infty \}.
\]

and is a closed convex set.

**Proposition 3** When the margin constraint (3) is imposed on the holdings of the investor, the effective domain is given by

\[
\mathcal{N} = \{ (a, b) \in \mathbb{R} \times \mathbb{R}^2, (1 - \lambda^+) a \leq b_i \leq (1 + \lambda^-) a, \} i = 1, 2 \}
\]

and \( e(a, b) \equiv 0 \), for all \( (a, b) \in \mathcal{N} \).

Following the derivation in Cuoco (1997), the optimization problem (4) is equivalent to

\[
F(W_t, Y_t) = \max_{(c \in C, (x,z) \in Q)} E_t \left[ \int_t^\infty u(c_s) e^{-\theta(s-t)} ds \right]
\]

\[
\pi_t W_t = E_t \left[ \int_t^\infty \pi_s (c_s - Y_s) ds \right]
\]

\[
dY_s = Y_s (mds + \Sigma^\top dw_s)
\]

\[
W_s > -K,
\]

with initial wealth and income given by \( W_t > 0, Y_t > 0 \). The condition \( W_t > -K \), for \( K > 0 \), rules out doubling strategies (see Harrison and Kreps (1979)). Denote by \( \tilde{u}(y,t) = \max_{c \geq 0} (u(c)e^{-\theta t} - yc) \) the convex conjugate of \( u(c)e^{-\theta t} \). As described in Cuoco (1997), to solve the optimization problem (4), it is enough to determine the saddle point \( (c^*, \psi^*, (a^*, b^*)) \) of the functional

\[
\mathcal{L}(c, \psi, (a, b)) = E_0 \left[ \int_0^\infty u(c_s) e^{-\theta s} ds \right] - \psi E_0 \left[ \int_0^\infty \pi_s (c_s - Y_s) ds \right] ,
\]

10
The maximization over $c$ yields

$$u'(c^*_s)e^{-\theta s} = \psi \pi_s.$$  

The Lagrange multiplier $\psi$ is determined by the budget constraint

$$E_0 \left[ \int_0^\infty \pi_s(I(\psi \pi_s e^{\theta s}) - Y_s) ds \right] = W_0,$$

where $I$ is the inverse of the marginal utility function. If we define a process $X$ by

$$X_t = \psi \pi_t e^{\theta t},$$

the dual problem is expressed as

$$J(X_0, Y_0) = \min_{(a,b) \in \mathcal{N}} E_0 \left[ \int_0^\infty (\tilde{u}(X_s) + X_s Y_s) e^{-\theta s} ds \right].$$  

(6)

For the case of a CRRA investor the indirect utility function starting at time $t$ is defined as

$$J(X_t, Y_t) = \begin{cases} \min_{(a,b) \in \mathcal{N}} E_t \left[ \int_t^\infty \left( \frac{\gamma X_{s-1}}{1-\gamma} + X_s Y_s \right) e^{-\theta(s-t)} ds \right] , & \gamma \neq 1, \\ \min_{(a,b) \in \mathcal{N}} E_t \left[ \int_t^\infty (-\ln X_s + X_s Y_s) e^{-\theta(s-t)} ds \right] , & \gamma = 1 \end{cases}$$

where

$$dX_s = X_s \left( (\theta - r - a_s) ds - (\sigma^{-1} (b_s - a_s I + \mu - r I))^\top dw_s \right)$$

$$dY_s = Y_s (mds + \Sigma^I dw_s).$$

Properties for the Dual Function $J$. As explained in He and Pages (1993), $J$ is non-increasing and strictly convex in $X$. It is also easy to see that $J$ is non-decreasing and concave in $Y$. Moreover, primal and dual variables are linked by the following relationships

$$W = -J_1 \text{ and } X = F_1.$$  

(7)

Finally, dropping the time index $t$, $J$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\theta J = \gamma X \frac{\gamma - 1}{1 - \gamma} + XY + (\theta - r)XJ_1 + mYJ_2 + \frac{Y^2}{2J_1} \sum_{i=1}^{2} (\sigma_i \Sigma_{i} J_{i2}) + \frac{Y^2}{2} \sum_{i=1}^{2} \Sigma_{i}^2 J_{i2}$$

$$+ \min_{(a,b) \in \mathcal{N}} \left\{ -aXJ_1 + \frac{X^2}{2} \sum_{i=1}^{2} \left( \frac{b_i - a + \mu_i - r - \sigma_i \Sigma_{i} J_{i2}}{\sigma_i^2} \right)^2 J_{i1} \right\} , \quad \gamma \neq 1.$$
Conditions on Parameters. In addition to conditions imposed earlier, we will impose that the following parameters $A, B, C$ are positive. These parameters appear in the characterization of the value function, the optimal consumption plan and the optimal investment strategy in both the optimization problem of an investor that does not receive income, as well as in the optimization problem of an investor that receives income.

\[
A = \frac{1}{\frac{\theta}{\gamma} + \frac{\gamma - 1}{\gamma}(r + \frac{1}{2\gamma} \sum_{i=1}^{2} (\frac{\mu_i - r}{\sigma_i})^2)} > 0
\]

\[
B = \frac{1}{r - m + \sum_{i=1}^{2} \frac{\Sigma_i(b_i - a + \mu_i - r)}{\sigma_i}} > 0
\]

\[
C = \theta + (\gamma - 1)(m - \gamma \sum_{i=1}^{2} \frac{\Sigma_i^2}{2}) > 0.
\]

2.2 No Income

In this section we review some of the results discussed in Cuoco and Liu (2000) for an investor that is subject to margin requirements and that does not receive income; i.e., $Y \equiv 0$. We are particularly interested in the situation when the investor that is not subject to a margin requirement chooses an allocation that would violate the margin requirement of the constrained investor. This condition can be written as

\[
\sum_{i=1}^{2} \frac{\mu_i - r}{\gamma \sigma_i^2} \lambda_i > 1, \tag{8}
\]

where $\lambda_i$ are the margin coefficients, and depend on whether the investor holds the asset long (in which case $\lambda_i = \lambda^+$), or short (when $\lambda_i = -\lambda^-$).

The dual value function satisfies the ordinary differential equation

\[
\theta J(X) = \min_{(a,b) \in \cal{N}} \frac{\gamma X^{\frac{\gamma - 1}{\gamma}}}{1 - \gamma} + (\theta - r - a) X J'(X) + \frac{X^2}{2} \sum_{i=1}^{2} \frac{(b_i - a + \mu_i - r)^2}{\sigma_i^2} J''(X).
\]

The results of Cuoco and Liu (2000) imply the following: \footnote{Cuoco and Liu (2000) provide the general framework for finding the optimal investment and consumption strategy and examples with one riskless and one risky asset. We extend their work by providing an explicit description of the optimal investment strategy for the case of two risky assets.}

**Proposition 4** When both risky assets have positive Sharpe ratios, the optimal allocations are given
Proposition 5  If asset 1 has a positive Sharpe ratio, \( \mu_1 > r \), and asset 2 has a negative Sharpe ratio, \( \mu_2 < r \), the optimal allocations are given by

\[
\frac{z_1}{W} = \begin{cases} 
\frac{1}{\lambda^+} \left( \mu_1 - \mu_j + \frac{\gamma \sigma_j^2}{\lambda^+} \right), & -\frac{\gamma \sigma_j^2}{\lambda^+} \leq \mu_i - \mu_j \leq \frac{\gamma \sigma_j^2}{\lambda^+}, \\
0, & \mu_i \leq \mu_j - \frac{\gamma \sigma_j^2}{\lambda^+}, \\
\frac{1}{\lambda^+}, & \mu_j \leq \mu_i - \frac{\gamma \sigma_i^2}{\lambda^+},
\end{cases}
\]

\[
\frac{z_2}{W} = \begin{cases} 
\frac{1}{\lambda^-} \left( \mu_2 - \mu_j + \frac{\gamma \sigma_j^2}{\lambda^-} \right), & -\frac{\gamma \sigma_j^2}{\lambda^-} \leq \mu_1 - \mu_j \leq \frac{\gamma \sigma_j^2}{\lambda^-}, \\
0, & \mu_1 \leq \mu_j - \frac{\gamma \sigma_j^2}{\lambda^-}, \\
-\frac{1}{\lambda^-}, & \mu_j \leq \mu_1 - \frac{\gamma \sigma_1^2}{\lambda^-},
\end{cases}
\]

where \( i, j = 1, 2, i \neq j \).

The proofs of the propositions are given in Appendix A.

Propositions 4 and 5 show that the optimal asset allocations do not depend on the level of wealth. In addition, the margin requirement influences both the amounts purchased of each risky asset, as well as whether an asset is held at all or not. For example, we note that if margin requirements for long positions are larger (\( \lambda^+ \) is close to one), it becomes more likely that the investor holds one, rather than both, risky assets. This result indicates that a mutual fund theorem does not hold when the investor is subject to margin requirements. The intuition is that, the investor chooses his allocation to reflect the opportunity cost associated with margin requirements; effectively holding risky assets is costly in the sense of tying up capital.

2.3 The case with Income

The case of an investor that receives an income stream and is subject to margin requirements is considerably more complicated than the case without income. Intuitively, from the work of Merton (1971), we know that without margin requirements, the investor should discount his future earnings, add the discounted value to current wealth, and make an investment choice based on the total, effective,
wealth. Since future earnings may be a significant portion of the effective wealth (and possibly many
times the current wealth), the allocation may violate the margin constraint. In addition, the extent
to which the margin constraint is binding depends on the ratio between the current wealth and the
discounted value of future earnings. From the results in Section 2.2., we anticipate that this leads to
asset substitution between the risky assets, especially when the constraint is most binding; i.e., when
the ratio of current wealth to discounted future earnings is small.

To study the impact of the income stream on the asset allocations, we choose parameters that
satisfy

\[ \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i < \sum_{i=1}^{2} \frac{\mu_i - r}{\gamma \sigma_i^2} \lambda_i < 1. \]

The inequality on the right ensures that the margin requirement for the optimal allocations in the
problem without income is less than the wealth, and thus non-binding. The inequality on the left
implies that income generates a positive net demand for risky assets. We show below, in our review
of the case without margin requirements, that if the inequality on the left is not satisfied, the hedging
demand for the income stream would actually reduce the total allocation in the risky assets.

### 2.3.1 Case without Margin Requirements

In the case that the investor does not face margin requirements the optimal asset allocations \( z_i^f, i = 1, 2 \)
and optimal consumption \( c \) are given by

\[
\begin{align*}
    z_i^f &= \frac{\mu_i - r}{\gamma \sigma_i^2} W + B \left( \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i} \right) Y \\
    c^f &= \frac{W + BY}{A}.
\end{align*}
\]

### 2.3.2 Case with Margin Requirements

We define by \( y \) the inverse relative risk aversion of the dual value function \( J \)

\[
y = -\frac{J_1}{XJ_{11}} > 0.
\]

Since primal and dual variables are linked by relationships (7), it follows that

\[
y = -\frac{J_1}{XJ_{11}} = -\frac{WF_{11}}{F_1},
\]
i.e., $y$ is equal to the relative risk aversion of the primal value function. In the case without margin constraints, we have $y \equiv \gamma$. In Appendix A we show that the substitutions

$$
\begin{align*}
\gamma & \rightarrow y \\
\mu_i - r & \rightarrow \mu_i - r + \sigma_i \Sigma_i (y - \gamma), \ i = 1, 2,
\end{align*}
$$

reduce the problem with income to the problem without income, studied in Section 2.2. This analogy allows us to characterize the optimal investment strategy of the investor in terms of $y$, or, equivalently, in terms of the ratio, $v = \frac{W}{Y}$, of current wealth to income.

**Proposition 6** There are three distinct regions in terms of the relative risk aversion $y$, or the current wealth to income ratio $v$, regarding the behavior of the investor. In the first region, for large values of the current wealth to income ratio $v$, where the relative risk aversion $y$ satisfies

$$
\gamma < \left(1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i\right)^{-1} \gamma \sum_{i=1}^{2} \left(\frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right) \lambda_i < y,
$$

the investor holds both risky assets, and the margin requirement is not binding. The investor’s risky asset allocation is smaller than the allocation of an investor with the same wealth and income that does not face margin requirement. At the frontier between the first and the second regions, the fraction of wealth invested in risky asset $i$ is given by

$$
\frac{z_i}{W} = \frac{\mu_i - r - \Sigma_i}{\gamma \sigma_i^2} + \left(\frac{\Sigma_i}{\sigma_i} \gamma \sigma_j^2 - \frac{\Sigma_i}{\sigma_j} \gamma \sigma_i^2\right) \lambda_j \left(1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i\right)^{-1} \gamma \sum_{i=1}^{2} \left(\frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right) \lambda_i.
$$

The second region is defined by values of the current wealth to income ratio $v$ such that the relative risk aversion $y$ satisfies

$$
\max \ \{y^*_i, y^*_j\} < y < \left(1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i\right)^{-1} \gamma \sum_{i=1}^{2} \left(\frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right) \lambda_i,
$$

with

$$
y^*_k = \frac{\lambda_k(\mu_k - r - \gamma \sigma_k \Sigma_k) - \lambda_k(\mu_h - r - \gamma \sigma_h \Sigma_k)}{\lambda_h \sigma_k^2 + \lambda_k(\lambda_k \sigma_h \Sigma_k - \lambda_h \sigma_k \Sigma_h)}, \ k, h = 1, 2, k \neq h.
$$

In this region, the investor holds both assets in proportions that depend on the current wealth to income ratio, and the margin constraint is binding. The third region corresponds to small values of the current wealth to income ratio $v$ and is defined by

$$
0 \leq y \leq \max \{y^*_i, y^*_j\}.
$$
In this region, the investor holds only one asset to the maximum extent allowed by the margin requirement.

The proof of Proposition 6 is provided in the appendix.

Proposition 6 indicates that the investor engages in asset substitution as the margin constraint becomes binding.\(^9\) We note that even though the investor shifts from a diversified portfolio to a less diversified one, he does not actually undertake more risk, since the allocation to the risky assets is bounded by the margin requirement. Intuitively, the investor tries to improve his return, within the bounds of the margin requirement imposed on him, and, in doing so, shifts his portfolio composition toward fewer assets.

The formulation of the different regions of investor behavior can also be interpreted in terms of the investor’s effective risk aversion \(y\). Since \(y\) measures risk aversion with respect to the investor’s value function, it captures the effect income has on risk aversion. The different regions correspond to high effective risk aversion when income contributes relatively little to the investor’s wealth; medium levels of effective risk aversion when the allocation to risky assets is binding but the investor still holds all the risky assets; and, low levels of effective risk aversion, when effective wealth is mostly due to future earnings. An intuitive interpretation of the investor’s behavior is that the investor appears less risk averse for low values of the ratio and more risk averse for high values.

**Properties of the Consumption and Investment Plans.** We can characterize optimal consumption using the following propositions.

**Proposition 7** The optimal consumption is increasing in current wealth and current income.

**Proposition 8** In the limit of zero current wealth, the optimal consumption rate is equal to the income rate.

**Deterministic Income.** In the case where income is deterministic, we can simplify some of the calculations. Under the condition

\[
\frac{\mu_2 - r}{\lambda_2} > \frac{\mu_1 - r}{\lambda_1},
\]

\(^9\)Cuoco and Liu (2004) find, in the context of a financial institution that needs to follow VaR reporting rules and that tries to optimize its asset selection, risk shifting behavior similar to the one we describe in this paper. This behavior leads the financial institution to invest in undiversified portfolios as the VaR constraint becomes binding.
we have that, in the region where the margin constraint is not binding, the reduced value function, \( f \), and the asset allocations \( z_1, z_2 \), satisfy

\[
(\theta + (\gamma - 1)m) f(v) = \frac{\gamma (f'(v))^{\gamma-1}}{1 - \gamma} + f'(v) + (r - m)v f'(v) - \frac{1}{2\gamma} \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\sigma_i} \right)^2 \frac{(f'(v))^2}{f''(v)}
\]

\[
\frac{z_i}{W} = -\frac{\mu_i - r}{\sigma_i^2} \frac{f'(v)}{v f''(v)}, \quad i = 1, 2.
\]

We note that the ratio of the risky asset allocations is constant, and equal to the case of an investor that does not face margin requirements. In the region where the investor holds both assets and the margin constraint is binding, the asset allocations satisfy

\[
\frac{z_i}{W} = \frac{\lambda_i \sigma_j^2}{\lambda_j^2 \sigma_i^2 + \lambda_i^2 \sigma_j^2} + \frac{\lambda_j (\lambda_i (\mu_j - r) - \lambda_j (\mu_i - r)) f'(v)}{\lambda_j^2 \sigma_i^2 + \lambda_i^2 \sigma_j^2} \frac{f'(v)}{v f''(v)}, \quad i, j = 1, 2, i \neq j,
\]

and the reduced value function \( f \) is the solution of

\[
(\theta + (\gamma - 1)m) f(v) = \frac{\gamma (f'(v))^{\gamma-1}}{1 - \gamma} + f'(v) + (r - m + \frac{\gamma \prod_{i=1}^{2} \sigma_i^2 \sum_{i \neq j}^{2} \frac{\mu_i - r}{\gamma \sigma_i^2}}{\sum_{i=1}^{2} \sigma_i^2 \lambda_j^2}) v f'(v)
\]

\[
+ \frac{\prod_{i=1}^{2} \sigma_i^2}{2 \sum_{i \neq j}^{2} \sigma_i^2 \lambda_j^2} v^2 f''(v) - \frac{1}{2\sigma_j^2 \lambda_j^2} \left( (\mu_2 - r) \lambda_1 - (\mu_1 - r) \lambda_2 \right)^2 \frac{(f'(v))^2}{f''(v)}.
\]

In the region where the investor holds just a single asset, we have

\[
0 \leq -\frac{vf''(v)}{f'(v)} \leq \frac{\lambda_i^2}{\sigma_i^2} \left( \frac{\mu_i - r}{\lambda_i} - \frac{\mu_1 - r}{\lambda_1} \right),
\]

and the optimal allocations are

\[
\frac{z_1}{W} = 0
\]
\[
\frac{z_2}{W} = \frac{1}{\lambda_2}.
\]

The function \( f \) satisfies

\[
(\theta + (\gamma - 1)m) f(v) = \frac{\gamma (f'(v))^{\gamma-1}}{1 - \gamma} + f'(v) + (r - m + \frac{\mu_2 - r}{\lambda_2}) v f'(v) + \frac{\sigma_2^2}{2 \lambda_2^2} v^2 f''(v).
\]

The boundary condition, at \( v = 0 \), is given by \( f(0) = ((1 - \gamma)(\theta + (\gamma - 1)m))^{-1} \) (see Appendix A).

We note that the choice on the “best” asset to hold, when the investor chooses to hold a single asset depends entirely on the expected return of each asset relative to the margin requirement for holding the asset, rather than on the Sharpe ratios. This is in line with the intuition that the investor is trying to increase the return of the portfolio as the margin constraint becomes more binding.
3 Numerical Examples

We performed numerical simulations for a base case of parameter values and monitored the optimal asset allocations in the risky and riskless assets. The base case parameters for the simulation are reported in Table I. The parameters correspond to the case of an investor that receives a deterministic income stream, and has two, independent, risky assets available, both with positive Sharpe ratios. The simulations were performed in Mathematica.

To calculate the value function and determine the thresholds when the investor’s asset allocation behavior changes, we follow an iterative procedure. Since, for large values of \( v \), we know the value function up to a constant, we arbitrarily chose the transition value \( v_1^* \) between the region where the margin constraint does not bind and the region where the margin constraint binds and the investor holds both assets. Effectively the choice of \( v_1^* \) completely specifies the value function, since, given \( v_1^* \), we are able to compute the value of \( f(v_1^*) \) and \( f'(v_1^*) \), and use them to integrate within the binding region until the value \( v_2^* \) where the investor only holds a single asset, and, similarly, integrate within the region where the margin constraint is binding and the investor holds a single asset up to \( v = 0 \). The goal is to match the boundary condition \( f(0) = 1/(1 - \gamma)(\theta + (\gamma - 1)m) \), which can be achieved by iteratively searching for the correct value of \( v_1^* \), until the desired match for the value of \( f(0) \) is achieved.

3.1 Measures of Risk and Diversification

Calvet, Campbell, and Sodini (2006) present an empirical analysis of diversification of household portfolios in Sweden, and describe several measures that quantify the degree that investors deviate from mean-variance optimal portfolios. We use the same measures in order to determine the potential magnitude of the impact of the financial constraints on diversification. We present the measures below, following the description in Calvet, Campbell, and Sodini (2006).

Denoting by \( r_{h,t}, r_{B,t} \) the returns of the risky asset portfolios of the constrained and unconstrained investors, respectively, we have the following variance decomposition

\[
r_{h,t} = \alpha_h + \beta_h r_{B,t} + \epsilon_{h,t},
\]

and, if we denote by \( \sigma_B, \sigma_h \) the standard deviation of the returns of the portfolio of the unconstrained and constrained investors respectively, we have

\[
\sigma_h^2 = \beta_h^2 \sigma_B^2 + \sigma_{\epsilon, h}^2.
\]
The interpretation of this decomposition is that the portfolio of the constrained investor has *systematic risk* $|\beta_h|\sigma_B$ and *idiosyncratic risk* $\sigma_{i,h}$. The *idiosyncratic variance share* is given by

$$\frac{\sigma^2_{i,h}}{\sigma^2_h} = \frac{\sigma^2_{i,h}}{\beta_h^2\sigma^2_B + \sigma^2_{i,h}}.$$

Another measure of portfolio diversification is the Sharpe ratio of the risky portion of the portfolio. We denote the Sharpe ratio of the portfolio of an investor that does not face financial constraints $S_B$, and the Sharpe ratio of a constrained investor $S_h$. These ratios are defined by the ratio of the excess return of the respective portfolio to the standard deviation of excess returns

$$S_h = \frac{\mu_h}{\sigma_h},$$

where $\mu_h, \sigma_h$, are the excess return and standard deviation of excess return for the portfolio of the constrained investor. The *relative Sharpe ratio loss* is defined by

$$RSRL_h = 1 - \frac{S_h}{S_B}.$$

While the relative Sharpe ratio loss is a measure of the diversification loss in the risky asset portion of the portfolio, it does not necessarily reflect the overall efficiency loss in the portfolio. To capture this loss, we define the *return loss* as the average return loss by the investor by choosing a suboptimal portfolio

$$RL_h = w_h(\mu_h - \mu_h),$$

where $w_h$ is the portion of the portfolio invested in risky assets.

Finally, we define a measure associated with utility losses for the constrained investor, compared to the unconstrained one, termed by Calvet, Campbell, and Sodini (2006). It is defined as the increase in the risk-free rate that would make the constrained investor indifferent between being constrained with the higher risk-free rate and being unconstrained. In the case of a risk-averse investor with CRRA preferences with risk aversion coefficient $\gamma$, Calvet, Campbell, and Sodini (2006) calculate the utility loss from the relationship

$$UL_h = S_B^2 - S_h^2.$$  

### 3.2 Base Case

For our base case parameter values we have chosen the returns and volatilities of the risky assets to be such that an unconstrained investor would invest a significant percentage of his wealth in them.
The expected return of the first asset over the risk-free rate is 3%, with volatility of 15%, while for the second asset, the expected return is 6% over the risk-free rate, with volatility of 20%. These parameters, together with a risk aversion coefficient of 3 result in an allocation of 44% in the first asset and 50% in the second asset for an investor that does not face margin requirements.

Table II examines the optimal asset allocations in the risky assets, and the different diversification measures for an investor that faces margin requirements. Since both assets have positive Sharpe ratios and are independent, the investor never shorts any of the risky assets. Since the margin requirement for long positions is 50%, the maximum total allocation in the risky assets can at most be 200% of current wealth. The allocations and the diversification measures are presented as functions of the ratio of current wealth to wealth from income.\(^{10}\) For large values of the ratio; i.e., when current wealth is much larger than wealth from income, the optimal allocation approaches the allocation of an investor that does not face a margin requirement. As the ratio of current wealth to wealth from income decreases, the allocation to risky assets increases as a percentage of wealth.

The margin requirement becomes binding for current wealth to wealth from income ratios below 29%. Below that value the investor shifts his portfolio towards asset 2, which has a higher expected return. Eventually, at current wealth to wealth from income ratios below 3%, the investor only holds asset 2. We point out that the diversification measures indicate that the portfolio deviates significantly from the benchmark of the portfolio of the unconstrained investor, similar to the empirical observations in Calvet, Campbell, and Sodini (2006). Low values of the ratio current to effective wealth correspond to investors with initially low wealth, or investors with a large income stream.

3.3 Comparative Statics

Table III presents comparative statics with respect to the Sharpe ratio of asset 2. The parameter values are the same as the base case parameter values, other than the volatility of asset 2, which has increased from 20% to 40%. By increasing the volatility of the second asset, we have reduced its Sharpe ratio, which is now lower than the Sharpe ratio of asset 1 leading to a smaller allocation in asset 2 in the limit where the ratio current to effective wealth approaches 1. Since the risky assets in this case are not as attractive as the risky assets in the base case, the overall allocations are smaller, and the margin constraint binds at lower values of the current wealth to wealth from income ratios.

\(^{10}\)Since, in this example, income is deterministic, the discounted value of future earnings is simply given by \(\frac{Y}{r+m}\), where \(Y\) is the current income rate, \(r\) is the interest rate, and \(m\) is the growth rate of income.
Despite the lower Sharpe ratio, for values of the ratio of current to effective wealth below 0.5%, the investor still prefers asset 2, since it provides higher expected return. The diversification measures indicate that the investor’s portfolio deviates more from the portfolio of the unconstrained investor than in base case.

In Table IV we present comparative statics with respect to the margin requirement. The parameters are the same as in Table I, other than the margin requirement for long positions, which we set to $\lambda^+ = 100\%$. This value implies that the investor can not borrow against the value of risky holdings. In this case the margin constraint binds at much higher levels of the ratio of current to effective wealth, since the investor can only hold up to 100% of his wealth in risky assets. The allocation to risky assets becomes binding at the relatively higher ratio of current wealth to wealth from income of 95%, while for current wealth levels below 11% of the effective wealth, the investor only holds the second asset in his portfolio.

3.4 Effective Wealth Equivalents and Optimal Consumption

In addition to the utility loss $UL_h$, we can compute another utility-related measure of the constraint cost, that we term effective wealth equivalent. Since the value function is a function of the ratio of current to effective wealth, the effective wealth equivalent is defined as the additional effective wealth that would make the investor indifferent between facing margin requirements and not facing margin requirements. We express the effective wealth equivalent in terms of percentage of effective wealth; i.e., the extra percentage of current wealth and extra percentage of income that would make the investor indifferent. For the base case parameters values, listed in Table I, when the ratio current to effective wealth is equal to 75%, the effective wealth equivalent is approximately 3%. As the ratio becomes smaller, the constraint influences more the behavior of the investor, resulting in higher wealth equivalents. When the ratio of current to effective wealth is 50%, the effective wealth equivalent is approximately 10%, while when the constraint first binds (at a current to effective wealth ratio of 29%), the effective wealth equivalent reaches approximately 22%. At the limit of no current wealth, the impact is the strongest, with the investor requiring an additional 78% of income in order to be indifferent between facing the margin requirement and not facing it. These results are consistent with the investor’s optimal consumption plan, under which the constrained investor consumes significantly less than the unconstrained one. For example, the constrained investor consumes 3.5% less when the ratio of the current to effective wealth is 75%, 13% less when the ratio is 50%, and 30% less when the
ratio is 29%. The magnitude of the effective wealth equivalent, and the drop in consumption, indicate the importance the margin constraint has in the investor’s behavior.

3.5 Discussion

The results we have presented in this section indicate that financial constraints can be a significant determinant of individual portfolios, and can, to some extent, account for empirical findings. For the case of deterministic income, the variable that is instrumental in the determination of the portfolios, and the extent to which they deviate from diversified portfolios, is the ratio of current wealth to discounted future wealth from income. For large values of this ratio the investor is largely unconstrained, while the constraint has the largest effect at low values of the ratio.

Although several of our assumptions are made for tractability reasons; e.g., infinite time horizon, infinite income stream, constant opportunity set, constant and known parameters of the stochastic price processes for the price of the risky assets, among others, one can still try to extrapolate from our results to observed individual investor behavior. Since deviations from diversified portfolios are understood in terms of the extent to which the financial constraint is binding, which in turn is measured in terms of the ratio of current wealth to future discounted wealth from income, young investors are most likely to be affected. As investors age, they accumulate wealth and they have a smaller remaining income stream, meaning they are less likely to be constrained, and more likely to hold portfolios that are close to be diversified. This prediction is in line with several empirical papers. For example, Goetzman and Kumar (2005) and Calvet, Campbell, and Sodini (2006) report that age is a significant determinant of under-diversification. Kumar (2005) reports that young investors are more likely to hold stocks with lottery-like payoffs that seemingly expose them to uncompensated risk. Our results also indicate that investors with relatively little current wealth and who are far from retirement would optimally hold more under-diversified portfolios. Goetzman and Kumar (2005) report that households that only have a retirement investment account, which presumably includes households that do not have enough wealth for an additional investment account, hold more under-diversified portfolios.

Our findings also provide a rational explanation for the empirical finding that investors only hold a small number of stocks in their portfolio: similar to Black (1972), constrained investors try to increase their expected return at the cost of holding less diversified portfolios by shifting toward portfolios with higher $\beta$.\textsuperscript{11} Ivkovic, Sialm, and Weisbenner (2004) show that, while investors hold relatively few

\textsuperscript{11}An interesting question is whether the inclusion of put options, with their higher leverage, would alleviate the
stocks in their portfolios, the number increases with an increase in account balance, which can be thought of as a proxy of current wealth.

While our findings indicate a clear link between financial constraints and under-diversified portfolios, the magnitude of the effect will depend on the particular situation. Alternative, behavioral based, explanations may also have significant effects, and it would be important to find variables that can distinguish between the two.

4 Conclusion

We have presented a characterization of the optimal investment strategy for an investor that faces margin requirements and who receives income that is spanned by the risky assets in the financial markets. We found that the inability of the investor to trade his future earnings, coupled with margin requirements on his investment on risky assets, lead him to deviate from the diversified portfolio of the unconstrained investor, and hold fewer assets as the constraint becomes more binding. The assets favored by the constrained investor are the ones that provide higher expected returns, even at the expense of higher volatility. Although we concentrate on the case of two, independent, risky assets, our analysis can be extended to the case of more assets.

An interesting extension of our work would be to consider assets with different margin requirements. In this case, we expect that the assets that have the highest return, when leveraged to the greatest extent possible, would appear most attractive to constrained investors. Such behavior would be in line with the preference of individual investors for residential real estate investments over financial investments, due to the lower margin requirements for residential real estate.

Our work can potentially have implications beyond the partial equilibrium setting and help understand the cross-sectional Sharpe ratios observed in the stock markets. Constantinides, Donaldson, and Mehra (2002) provide a general equilibrium model with different types of investors and argue that the presence of investors who face borrowing constraints, margin requirements, and who cannot trade their future earnings\textsuperscript{12}, increases the risk premium for the single risky asset in their model.\textsuperscript{13} They

\textsuperscript{12}Detemple and Serrat (2003) show that preventing agents from borrowing against their future income alone can help to resolve the riskfree rate puzzle but cannot account for the magnitude of the Sharpe ratio.

\textsuperscript{13}See also the paper by Basak and Cuoco (1998).
suggest that borrowing constraints can play a significant role in understanding the size of the historical equity premium in the US equity markets. While our paper only addresses the case of partial equilibrium, it indicates that it would be interesting to extend the model of Constantinides, Donaldson, and Mehra (2002) to multiple risky assets, and study the cross-sectional implications. Based on our results, one would expect that, since constrained investors prefer assets with high absolute expected returns, even with high volatility, the equilibrium Sharpe ratios for such assets may be lower than for assets with lower expected returns. Such an implication would also weaken the relationship predicted by the Capital Asset Pricing Model between the covariance of a risky asset with the market portfolio and the asset’s expected return. While it is unclear what the magnitude of such an effect would be, the problem can be studied both theoretically and empirically, and would be an interesting direction for future research.
5 Appendix A

Two period model: Section I

We will provide the results in Section I for each of the different regions presented in Figure 1.

**Non binding Region.** When wealth is high relative to income, the margin constraint is not binding. The optimal consumption in the second period $c_1$, is related to the optimal consumption in the first period $c_0$, through the first order conditions

\[
\frac{1}{3} \left( \frac{1.891}{c_{11}} + \frac{0.724}{c_{12}} + \frac{0.535}{c_{13}} \right) = \frac{1}{c_0},
\]

\[
\frac{1}{3} \left( \frac{0.975}{c_{11}} + \frac{1.422}{c_{12}} + \frac{0.693}{c_{13}} \right) = \frac{1}{c_0},
\]

\[
\frac{1}{3} \left( \frac{1}{c_{11}} + \frac{1}{c_{12}} + \frac{1}{c_{13}} \right) = \frac{1}{c_0},
\]

which can be solved to give

\[
c_{11} = 1.11c_0
\]

\[
c_{12} = 1.09c_0
\]

\[
c_{13} = 0.84c_0,
\]

where $c_{11}, c_{12}, c_{13}$ are the optimal consumptions in the second period in states 1, 2, 3 respectively. Given the optimal consumption in the first period, $c_0$, the asset allocations are

\[
z_1 = 0.13c_0
\]

\[
z_2 = 0.31c_0
\]

\[
x + Y = 0.56c_0,
\]

and, using the budget constraint in the first period leads to

\[
c_0 = 0.5(W + Y).
\]

For the allocations to satisfy the margin constraint, we have

\[
z_1 + z_2 \leq W - c_0 \Rightarrow \frac{W}{Y} \geq 2.57.
\]

Thus, the investor follows the same investment and consumption strategy as the unconstrained investor, when wealth is above 2.57 times income, or equivalently when the wealth to effective wealth ratio is above 72%.
Binding Region with Two Assets. When wealth is less than 2.57 times income, the margin constraint is binding

\[ z_1 + z_2 = W_0 - c_0. \]

From the budget constraint at time 0, we have that the investment in the riskless asset is zero, \( x = 0 \).

Hence the investor maximizes

\[
\max_{z_1, z_2 \geq 0} \left[ \ln(W - z_1 - z_2) + \frac{1}{3} \left( \ln(1.891 z_1 + 0.975 z_2 + Y) + \ln(0.724 z_1 + 1.422 z_2 + Y) + \ln(0.535 z_1 + 0.693 z_2 + Y) \right) \right]
\]

The first order conditions are

\[
\frac{1}{3} \left( \frac{1.891}{c_{11}} + \frac{0.724}{c_{12}} + \frac{0.535}{c_{13}} \right) = \frac{1}{c_0},
\]

\[
\frac{1}{3} \left( \frac{0.975}{c_{11}} + \frac{1.422}{c_{12}} + \frac{0.693}{c_{13}} \right) = \frac{1}{c_0}.
\]

Binding Region with One Asset. While we cannot solve the previous system of equations in closed form for \( z_1, z_2 \), it is possible to determine that there is a threshold value for the ratio of wealth over income, for which the allocation in the second risky asset is zero. This threshold can be determined by

\[ z_2 = 0 \]

\[ z_1 = W - c_0, \]

implying that the allocation in the riskless asset is zero, \( x = 0 \), and

\[ c_{11} = Y + 1.891(W - c_0) \]

\[ c_{12} = Y + 0.724(W - c_0) \]

\[ c_{13} = Y + 0.535(W - c_0), \]

which, combined with the first order conditions gives the following 2 \( \times \) 2 system for \( c_0, \frac{W}{W + Y} \)

\[
\frac{1.891}{Y + 1.891(W - c_0)} + \frac{0.724}{Y + 0.724(W - c_0)} + \frac{0.535}{Y + 0.535(W - c_0)} = \frac{3}{c_0},
\]

\[
\frac{0.975}{Y + 1.891(W - c_0)} + \frac{1.422}{Y + 0.724(W - c_0)} + \frac{0.693}{Y + 0.535(W - c_0)} = \frac{3}{c_0}.
\]

We can solve this system numerically, to determine the value of the threshold and the value of consumption at the threshold

\[
\frac{W}{W + Y} = 52\% \]

\[ c_0 = 0.94W. \]
We note that as the ratio of wealth over income decreases, the investor shifts his portfolio towards the first asset. The intuition behind this substitution is that the investor is constrained from choosing a portfolio with enough exposure to the risky assets and trades the higher return of the first risky asset against the diversification benefit from holding both assets in the proportions chosen by the unconstrained investor.

**No Financial Holdings.** It is possible to show that there exists an additional threshold in the ratio of wealth over income, below which the investor optimally chooses to consume his entire initial wealth, and does not participate in the financial markets. Indeed, when \( \frac{W}{W + Y} \leq 52\% \), the allocation in the first asset is given by

\[
z_1 = W - c_0,
\]

and the investor maximizes

\[
\max_{z_1 \geq 0} \left[ \ln(W - z_1) + \frac{1}{3} (\ln(1.891z_1 + Y) + \ln(0.724z_1 + Y) + \ln(0.535z_1 + Y)) \right].
\]

This maximization gives that, at a ratio of wealth to income equal to 0.95, the investment in the first asset becomes zero. At values of the ratio less than or equal to 0.95, the optimal consumption is equal to the initial wealth of the investor and the investor does not hold any assets.

**Discussion.** This example illustrates that as the wealth to income ratio decreases, the investor shifts his allocation and consumption strategies in the following way: for large values of the ratio (above 2.57), the margin constraint is not binding and he follows the same strategy as an unconstrained investor. As the ratio drops below 2.57 he shifts his risky asset allocation towards the first asset, which has a higher expected return. This shift results in a region, for values of the ratio between 1.09 and 0.95, where the investor only holds the first asset. As the ratio decreases, the investor consumes higher percentages of his initial wealth, resulting, at the point when wealth is equal to 0.95 times income, to a new threshold, below which all initial wealth is consumed in the first period and no investment is made.

**Proof of Proposition 1.** Assume that \((c, (x, z))\) is a feasible consumption and investment plan for initial conditions \((W_t, Y_t)\). Then, for all \(\alpha > 0\), we show that \((\alpha c, (\alpha x, \alpha z))\) is a feasible consumption and investment plan for initial conditions \((\alpha W_t, \alpha Y_t)\).

Consider the dynamics for the wealth process \(W_\alpha\), with initial conditions \((\alpha W_t, \alpha Y_t)\), following the consumption and investment plan \((\alpha c, (\alpha x, \alpha z))\). We have

\[
dW_{\alpha s} = \alpha W_s ds - \alpha c_s ds + \alpha Y_s ds + \alpha z_s^T (\mu - rT) ds + \alpha z_s^T \sigma dw_s = \alpha dW_s,
\]
therefore, $W_{\alpha s} = \alpha W_s$. Similarly, we have $Y_{\alpha s} = \alpha Y_s$. It is also easy to see that the investment strategy satisfies the margin requirement

$$\lambda^+([\alpha z_1]^+ + [\alpha z_2]^+) + \lambda^-([\alpha z_1]^- + [\alpha z_2]^-) = \alpha \left( \lambda^+(z_1^+ + z_2^+) + \lambda^-(z_1^- + z_2^-) \right) \leq \alpha W.$$

By contradiction, it is now easy to see that if $(c^*, (z_1^*, z_2^*))$ is an optimal investment strategy for initial conditions $(W_t, Y_t)$, then $(\alpha c^*, (\alpha z_1^*, \alpha z_2^*))$ is an optimal investment strategy for initial conditions $(\alpha W_t, \alpha Y_t)$. From the form of the utility function from consumption, we then have that

$$F(\alpha W, \alpha Y) = \alpha^{1-\gamma} F(W, Y).$$

**Proof of Proposition 2.** To show that $F$ is non-decreasing in $(W_t, Y_t)$, is simple, since, given an initial endowment $(W_t, Y_t)$, it is easy to see that starting with wealth $W_t' > W_t$ or income $Y_t' > Y_t$ at time $t$, the optimal strategy for the initial condition $(W_t, Y_t)$ is still admissible and potentially non-optimal for the problem with initial conditions $(W_t', Y_t')$. This implies that $F$ is non-decreasing in $W$ and $Y$. To show concavity, consider two initial conditions $(W_t, Y_t)$ and $(W_t', Y_t')$ and $\alpha \in (0, 1)$. Denote $(c, (x, z))$ and $(c', (x', z'))$ the optimal strategies respectively for the two initial conditions. Then, the strategy $S : (\alpha c + (1 - \alpha)c', \alpha x + (1 - \alpha)x', \alpha z + (1 - \alpha)z')$ is admissible for the initial condition $I : (\alpha W_t + (1 - \alpha)W_t', \alpha Y_t + (1 - \alpha)Y_t')$. Denoting $W_\alpha$ the wealth process associated with strategy $s$ and initial condition $I$, we have

$$W_{\alpha s} = \alpha W_s + (1 - \alpha)W'_s,$$

and similarly for the income process

$$Y_{\alpha s} = \alpha Y_s + (1 - \alpha)Y'_s.$$

It remains to check that the margin constraint is satisfied. To do so, we notice that the function $g$ defined by

$$g(x, z) = \lambda^+ \sum_{i=1}^{2} z_i^+ + \lambda^- \sum_{i=1}^{2} z_i^- - x - \sum_{i=1}^{2} z_i,$$

is convex. This implies that

$$g(\alpha x + (1 - \alpha)x', \alpha z + (1 - \alpha)z') \leq \alpha g(x, z) + (1 - \alpha)g(x', z') \leq 0,$$

which proves that the strategy $s$ satisfies the margin constraint. Finally, since the utility from consumption, $u$, is strictly concave we have

$$E_t \left[ \int_t^\infty u(\alpha c_s + (1 - \alpha)c'_s) e^{-\theta s} ds \right] > E_t \left[ \int_t^\infty (\alpha u(c_s) + (1 - \alpha)u(c'_s)) e^{-\theta s} ds \right],$$
which implies that
\[ F(\alpha W_t + (1 - \alpha)W'_t, \alpha Y_t + (1 - \alpha)Y'_t) > \alpha F(W_t, Y_t) + (1 - \alpha)F(W'_t, Y'_t). \]

**Proof of Proposition 3.** The relationship \( e(a, b) \equiv 0 \) comes from the fact that \( Q \) is a cone. Then, it is easy to see that we must have \( a \geq 0, b_i \geq 0, i = 1, 2 \). If \( z_i \geq 0, i = 1, 2 \) we have
\[- ax - \sum_{i=1}^2 b_i z_i = - a \left( x + (1 - \lambda) \sum_{i=1}^2 z_i \right) - \sum_{i=1}^2 (b_i - (1 - \lambda^+)a) z_i. \]
Since \( z_i \geq 0, i = 1, 2 \) we must have \( b_i - (1 - \lambda^+)a \geq 0, i = 1, 2 \). Similarly, when \( z_i \leq 0, i = 1, 2 \), we have
\[- ax - \sum_{i=1}^2 b_i z_i = - a \left( x + (1 + \lambda) \sum_{i=1}^2 z_i \right) - \sum_{i=1}^2 (b_i - (1 + \lambda^-)a) z_i. \]
Since \( z_i \leq 0, i = 1, 2 \), we must have \( b_i - (1 + \lambda^-)a \leq 0, i = 1, 2 \).

**Proof of Proposition 4.** We have that the dual value function \( J \) satisfies
\[ \theta J(X) = \min_{(a,b)\in\mathcal{N}} \frac{\gamma X^{\gamma-1}}{1 - \gamma} + (\theta - r - a)XJ'(X) + \frac{X^2}{2} \sum_{i=1}^2 \frac{(b_i - a + \mu_i - r)^2}{\sigma_i^2}J''(X). \]
The minimization leads to either
\[ (i) \quad a^* = b^*_i = 0, i = 1, 2 \]
\[ (ii) \quad -a^*J'(X) = 0, \quad b^*_i \in ((1 - \lambda^+)a^*, (1 + \lambda^-)a^*), i = 1, 2 \]
\[ (iii) \quad a^* = \frac{\prod_{i=1}^2 \sigma_i^2}{\sum_{i,j} \sigma_i^2 \lambda_j^2} \left( \sum_{i=1}^2 \frac{\mu_i - r}{\sigma_i^2} \lambda_i + \frac{J'(X)}{XJ''(X)} \right), \text{ if } a^* \geq 0 \]
\[ b^*_i \left\{ \begin{array}{l} = (1 - \lambda_i)a^*, \text{ if } \lambda_i \neq 0 \\ \in ((1 - \lambda^+)a^*, (1 + \lambda^-)a^*) \text{ otherwise}, \end{array} \right. \]
with \((\lambda_1, \lambda_2) \in \{0, \lambda^+, -\lambda^-\} \setminus (0,0)\). Given the structure of the ODE and the homogeneity of degree \( 1 - \gamma \) of the primal value function, we look for a solution of the type
\[ J(X) = \frac{\gamma K X^{\gamma-1}}{1 - \gamma}, \]
where \( K \) is a constant to be determined.\(^{14}\) This implies that either \( a^* = 0 \) or
\[ a^* = \frac{\gamma \prod_{i=1}^2 \sigma_i^2}{\sum_{i,j} \sigma_i^2 \lambda_j^2} \left( \sum_{i=1}^2 \frac{\mu_i - r}{\gamma \sigma_i^2} \lambda_i - 1 \right), \]
\(^{14}\)We assume that the parameters are such that \( K > 0 \). Since the constrained value function is lower than the unconstrained one, it is easy to see that \( K > A \) when \( \gamma > 1 \), and \( K < A \) when \( \gamma < 1 \).
so we can conclude that $a^*, b^*$ are constants. Performing the minimization leads to

$$
\left[ \frac{\theta}{\gamma} + \frac{(\gamma - 1)}{\gamma} \left( r + a^* + \frac{1}{2\gamma} \sum_{i=1}^{2} \frac{(b_i^* - a^* + \mu_i - r)^2}{\sigma_i^2} \right) \right] K = 1.
$$

It is easy to check that regardless the sign of $1 - \gamma$, the minimization is equivalent to the following static minimization

$$
\min_{(a, b) \in \mathcal{N}} a + \frac{1}{2\gamma} \sum_{i=1}^{2} \frac{(b_i - a + \mu_i - r)^2}{\sigma_i^2}.
$$

The wealth is given by

$$
W = -J'(X) = K X^{-\frac{1}{\gamma}},
$$

and for $i = 1, 2$ the optimal portfolio allocations and consumption are

$$
z_i = \frac{b_i^* - a^* + \mu_i - r}{\gamma \sigma_i^2},
$$

$$
c = \frac{W}{K}.
$$

Looking back at the different cases for $(a^*, b^*)$, we can exclude case (i), since it corresponds to the unconstrained solution which, by assumption, violates the margin constraint. Then, if the minimum is achieved at some interior solution, we must have

$$
b_i^* - a^* + \mu_i - r = 0, \quad i = 1, 2,
$$

and it is optimal to choose $a^* = \min(\frac{\mu_i - r}{\lambda^+}) \neq 0, i = 1, 2$. However, this implies that $z_i = 0, i = 1, 2$, and, since $a^* \neq 0$, the wealth must be zero, which cannot be optimal. Now assume that the minimum is achieved for $b_j^* = (1 + \lambda^-)a^*$ and $b_i^* = (1 - \lambda^+)a^*$ or $b_i^*$ interior. In this case, we must have $z_j \leq 0$. However, the optimal allocation $z_j$ is given by

$$
z_j = \frac{\lambda^- a^* + \mu_j - r}{\gamma \sigma_j^2} > 0,
$$

since $a^* \geq 0$, which leads to a contradiction. Thus $b_j = (1 + \lambda^-)a$, for some $a \geq 0$ is not admissible.

If the minimum is achieved for $b_i = (1 - \lambda^+)a, i = 1, 2$, then we must have

$$
a^* = \frac{\gamma \prod_{i=1}^{2} \sigma_i^2}{\lambda^+ \sum_{i=1}^{2} \sigma_i^2} \left( \sum_{i=1}^{2} \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{1}{\lambda^+} \right),
$$

and the minimum is

$$
\frac{1}{2\gamma} \left[ \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\sigma_i} \right)^2 - \frac{\prod_{i=1}^{2} \sigma_i^2}{\sum_{i=1}^{2} \sigma_i^2} \left( \frac{\gamma}{\lambda^+} - \sum_{i=1}^{2} \frac{\mu_i - r}{\sigma_i^2} \right)^2 \right].
$$
From the duality relationship, we can conclude that the constraint is binding with
\[
\lambda^+ \sum_{i=1}^{2} z_i = W.
\]
This implies that we must have \( z_i \geq 0, i = 1, 2 \). The condition is
\[
-\frac{\gamma \sigma^2_j}{\lambda^+} \leq \mu_i - \mu_j \leq \frac{\gamma \sigma^2_i}{\lambda^+}.
\]

It remains to examine the case where \( b_i = (1 - \lambda^+) a \) and \( b_j \) is interior. In this case, we must have
\[
b_j^* - a^* + \mu_j - r = 0,
\]
so in particular \( z_j = 0 \), which implies that
\[
a^* \geq \frac{\mu_j - r}{\lambda^+},
\]
and we are left to minimize
\[
\min_{a \geq \frac{\mu_j - r}{\lambda^+}} a + \frac{1}{2\gamma} \left( \frac{(-\lambda^+ a + \mu_i - r)^2}{\sigma^2_i} \right)
\]
We obtain
\[
a^* = \frac{\mu_i - \gamma \sigma^2_i}{\lambda^+} - \frac{\gamma \sigma^2_j}{(\lambda^+)^3}.
\]
However, for \( a^* \) to be admissible, we must have
\[
\mu_j \leq \mu_i - \frac{\gamma \sigma^2_j}{\lambda^+},
\]
otherwise we can choose \( a^* = 0 \), leading to \( b_i^* = 0, i = 1, 2 \) which is impossible. Finally, notice that we cannot have
\[
\mu_j \leq \mu_i - \frac{\gamma \sigma^2_i}{\lambda^+},
\]
at the same time. It follows that the minimum is
\[
\begin{cases}
\frac{\mu_j - r}{\lambda^+} - \frac{\gamma \sigma^2_j}{2(\lambda^+)^2} & \text{if } \mu_i - \mu_j \leq -\gamma \sigma^2_j \\
\frac{1}{2\gamma} \left[ \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\sigma_i} \right)^2 - \frac{2}{\sum_1^2 \sigma_i^2} \left( \sum_{i=1}^{2} \frac{\mu_i - r}{\sigma_i} \right)^2 \right] & \text{if } -\gamma \sigma^2_j \leq \mu_i - \mu_j \leq \gamma \sigma^2_j \\
\frac{\mu_i - r}{\lambda^+} - \frac{\gamma \sigma^2_i}{2(\lambda^+)^2} & \text{if } \gamma \sigma^2_j \leq \mu_i - \mu_j
\end{cases}
\]
The proof is complete. Proposition 5 can be proven similar to proposition 4.
Proof of Proposition 6. We use the dual approach to analyze the optimal consumption and asset allocation strategy. When there is no income the dual problem is

$$\min_{(a,b) \in \mathcal{N}} a + \frac{1}{2\gamma} \sum_{i=1}^{2} \frac{(b_i - a + \mu_i - r)^2}{\sigma_i^2}. \tag{6.1}$$

When income is not zero, we need to solve

$$\min_{(a,b) \in \mathcal{N}} -aX_J + \frac{X^2}{2} \sum_{i=1}^{2} \frac{(b_i - a + \mu_i - r - \sigma_i \Sigma_i \frac{Y J_{12}}{X J_{11}})^2}{\sigma_i^2} J_{11}. \tag{6.2}$$

Since

$$Y J_{12} = \gamma X J_{11} + J_1,$$

the problem becomes

$$\min_{(a,b) \in \mathcal{N}} -aX_J + \frac{X^2}{2} \sum_{i=1}^{2} \frac{(b_i - a + \mu_i - r - \gamma \sigma_i \Sigma_i - \sigma_i \Sigma_i J_{11} \frac{J_{1}}{X J_{11}})^2}{\sigma_i^2} J_{11}. \tag{6.3}$$

Let us denote

$$y = -\frac{J_1}{X J_{11}} > 0,$$

the inverse relative risk aversion of the dual function. Then the minimization problem is equivalent to

$$\min_{(a,b) \in \mathcal{N}} a + \frac{1}{2y} \sum_{i=1}^{2} \frac{(b_i - a + \mu_i - r + \sigma_i \Sigma_i (y - \gamma))^2}{\sigma_i^2}. \tag{6.4}$$

This problem is similar to the problem in the no income case by making the following substitutions

$$\gamma \rightarrow y$$

$$\mu_i - r \rightarrow \mu_i - r + \sigma_i \Sigma_i (y - \gamma), i = 1, 2.$$ 

The only additional case (with respect to the no income case analyzed before) to be considered is when the margin constraint is not binding. There are actually three regions to consider: the first one is the non-binding region, the second is the binding region with two assets held and finally the region where only one asset is held. We examine in details these regions.

Case 1: Non-Binding Region. In this case, we have $b_i^* = a^* = 0, i = 1, 2$. Taking into account the substitutions, it must be the case that

$$y > \sum_{i=1}^{2} \frac{\mu_i - r + \sigma_i \Sigma_i (y - \gamma)}{\sigma_i^2} \lambda_i,$$
or equivalently

\[ y > \frac{\gamma \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i} \right) \lambda_i}{1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i}. \]

**Case 2: Binding Region with Two Assets.** In this case, we have

\[
a^* = \frac{\prod_{i=1}^{2} \sigma_i^2}{\sum_{i \neq j} \sigma_j^2 \lambda_i^2} \left( \sum_{i=1}^{2} \frac{\mu_i - r}{\sigma_i^2} \lambda_i + \frac{1}{XJ_{11}} \left( J_1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i Y_{J12} \right) \right)
\]

\[
= \frac{\prod_{i=1}^{2} \sigma_i^2}{\sum_{i \neq j} \sigma_j^2 \lambda_i^2} \left( \gamma \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i} \right) \lambda_i - \left( 1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i \right) y \right)
\]

\[b^*_i = (1 - \lambda_i)a^*, \quad i = 1, 2.\]

Given the results from the no income case, we must have

\[ y < \frac{\gamma \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i} \right) \lambda_i}{1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i}, \]

and

\[-y \sigma_j^2 \leq \frac{\mu_i - r + \sigma_i \Sigma_i(y - \gamma)}{\lambda_i} - \frac{\mu_j - r + \sigma_j \Sigma_j(y - \gamma)}{\lambda_j} \leq \frac{-y \sigma_i^2}{\lambda_i^2}, \]

or, equivalently

\[
\text{max} \{ y^*_i, y^*_j \} < y < \left( 1 - \sum_{i=1}^{2} \frac{\lambda_i}{\sigma_i} \right)^{-1} \sum_{i=1}^{2} \frac{\mu_i - r - \sigma_i \Sigma_i \gamma}{\sigma_i^2} \lambda_i
\]

\[y^*_k = \frac{\lambda_k (\mu_k - r - \gamma \sigma_k \Sigma_k) - \lambda_k (\mu_h - r - \gamma \sigma_h \Sigma_h))}{\lambda_h \sigma_k^2 + \lambda_k (\lambda_k \sigma_h \Sigma_h - \lambda_h \sigma_k \Sigma_k)}, \quad k, h = 1, 2, k \neq h.\]

The point \( y^*_j \) where the investor optimally chooses to only hold asset \( j \) is such that \( z_i = 0 \). Therefore the condition for holding only asset \( j \) versus holding only asset \( i \) is \( y^*_j > y^*_i \).

**Case 3: Binding Region with Only Asset \( j \).** In this case, we have

\[ a^* = \frac{\sigma_j^2}{\lambda_j^2} \left( \frac{\mu_j - r}{\sigma_j^2} \lambda_j + \frac{\Sigma_j}{\sigma_j} (y - \gamma) \lambda_j - y \right) \]

\[b^*_i = a^* + \mu_i - r + \sigma_i \Sigma_i(y - \gamma) \]

\[b^*_j = (1 - \lambda_j)a^*.\]
The condition is

\[ 0 \leq y < y_j^*, \]

Note that it is not possible to move from the non-binding region into the binding region where only one asset is held without first going through the binding region where two assets are held.

**Dynamics of the Indirect Value Function \( J \).**

**Case 1: Non-Binding Region, \( b_i^* = a_i^* = 0, i = 1, 2 \).** In this case, the indirect value function \( J \) satisfies the following linear PDE

\[
\theta J = \frac{\gamma X^{\gamma-1}}{1-\gamma} + XY + (\theta - r)XJ_1 + mYJ_2 + \frac{1}{2} \sum_{i=1}^{2} \sigma_i^2 Y^2 J_{22} \\
- XY \sum_{i=1}^{2} \frac{\sum_i (\mu_i - r)}{\sigma_i} J_{12} + \frac{X^2}{2} \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\sigma_i} \right)^2 J_{11}.
\]

Given the homogeneity of the primal value function of degree \( 1 - \gamma \), we look for a solution of the type

\[
J(X, Y) = X^{\gamma-1} H(YX^{\frac{1}{\gamma}}).
\]

This implies that

\[
J_1(X, Y) = \frac{\gamma - 1}{\gamma} X^{-\frac{1}{\gamma}} H(YX^{\frac{1}{\gamma}}) + \frac{Y}{\gamma} H'(YX^{\frac{1}{\gamma}}) \\
J_2(X, Y) = X H'(YX^{\frac{1}{\gamma}}) \\
J_{12}(X, Y) = H'(YX^{\frac{1}{\gamma}}) + \frac{YX^{\frac{1}{\gamma}}}{\gamma} H''(YX^{\frac{1}{\gamma}}) \\
J_{11}(X, Y) = X^{-\frac{1}{\gamma} - 1} \left( (1 - \gamma)H(YX^{\frac{1}{\gamma}}) + (\gamma - 1)YX^{\frac{1}{\gamma}} H'(YX^{\frac{1}{\gamma}}) + Y^2 X^{\frac{2}{\gamma}} H''(YX^{\frac{1}{\gamma}}) \right) \\
J_{22}(X, Y) = X^{\frac{1}{\gamma} + 1} H''(YX^{\frac{1}{\gamma}}).
\]

We note that

\[
\gamma X J_{11}(X, Y) + J_1(X, Y) = Y J_{12}(X, Y).
\]

Given the properties of \( J \) with respect to \( Y \), it must be the case that \( H \) is increasing and concave. It follows that \( H \) must satisfy the following ODE

\[
\frac{1}{A} H(Z) = \frac{\gamma}{1-\gamma} + Z + \left( \frac{1}{A} - \frac{1}{B} \right) Z H'(Z) + \frac{1}{2} \sum_{i=1}^{2} \sigma_i^2 Z^2 H''(Z),
\]

with

\[
\sigma_{z_i} = \Sigma_i - \frac{\mu_i - r}{\gamma \sigma_i}, \quad i = 1, 2
\]
Since $J(X,0)$ is finite, the solution to the above ODE is given by

$$H(Z) = \frac{\gamma A}{1 - \gamma} + BZ - \frac{\gamma K}{\beta + \gamma - 1} Z^\beta,$$

where $K$ is a constant to be determined and $\beta$ is the positive root of the quadratic

$$\frac{1}{2} \sum_{i=1}^{2} \sigma_i^2 x^2 + \left( \frac{1}{A} - \frac{1}{B} - \frac{1}{2} \sum_{i=1}^{2} \sigma_i^2 \right) x = \frac{1}{A}$$

We have assumed $A > 0$ and $B > 0$, therefore, since $\frac{1}{A} - \frac{1}{B} < \frac{1}{A}$, we have $\beta > 1$. Hence

$$J(X,Y) = \frac{\gamma AX^{\frac{1 - \gamma}{\gamma}}}{1 - \gamma} + BYX - \frac{\gamma K}{\beta + \gamma - 1} X^\frac{\beta + \gamma - 1}{\gamma} Y^\beta$$

Note that if $K < 0$, for high values of $Y$, it could be the case that the indirect value function is larger than the unconstrained indirect value function $J_0(X,Y)$ given by

$$J_0(X,Y) = \frac{\gamma AX^{\frac{1 - \gamma}{\gamma}}}{1 - \gamma} + BYX.$$ 

This implies that we must have $K > 0$.

**Optimal Frontier.** At the frontier $Z^*$, we require continuity of the control $a^*$, which implies that $J$ is twice continuously differentiable, or, equivalently $H, H'$ and $H''$ must be continuous at $Z^*$. The boundary is such that

$$\sum_{i=1}^{2} \lambda_i z_i^* = W^*,$$

or, equivalently

$$y = \frac{\gamma \sum_{i=1}^{2} \left( \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i} \right) \lambda_i}{1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i}.$$ 

Given the results on the regions, it cannot be the case that the agent optimally chooses to enter into the binding region holding only one asset. Hence, at the boundary, we must have

$$a^* = \frac{\prod_{i=1}^{2} \sigma_i^2}{\sum_{i \neq j} \sigma_i^2 \lambda_i^2} \left( \sum_{i=1}^{2} \frac{\mu_i - r}{\sigma_i^2} \lambda_i + \frac{1}{X J_{11}} \left( J_1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i J_{12} \right) \right) = 0.$$
This implies that

\[ K(Z^\beta) = \frac{B \left( 1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i \right) (Z^* - Z)}{1 + (\beta - 1) \sum_{i=1}^{2} \frac{\mu_i - r}{\gamma \sigma_i^2} \lambda_i - \beta \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i} \]

\[ Z = \frac{1 - \frac{\mu_i - r}{\gamma \sigma_i^2} \lambda_i}{1 - \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i} \lambda_i} A > 0. \]  

(9)

Note that \( K > 0 \) implies that \( Z \leq Z^* \). Since we have assumed that when \( Y \) equals zero the margin constraint is not binding, the non-binding region is defined by

\[ \text{Non-binding region} = \{ (X,Y) \in \mathbb{R}_+^2, X^\frac{1}{\gamma} Y \leq Z^* \}. \]

**Optimal Allocations in the Non-Binding Region** \( X^\frac{1}{\gamma} Y \leq Z^* \)

Consumption, wealth and income are linked by the following relationship

\[ W = Ac - BY + Kc^{1-\beta} Y^\beta. \]  

(10)

One can notice that for the same \((W,Y)\), the consumption rate of the constrained investor, \( c \), is smaller than the consumption rate of the unconstrained investor \( c^f \). Using Itô’s lemma and identifying the coefficients with the dynamics of the wealth, the optimal portfolio allocations are given by

\[ z_i = \frac{\mu_i - r}{\gamma \sigma_i^2} W + B \left( \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i} \right) Y - \beta K \left( \frac{\mu_i - r}{\gamma \sigma_i^2} - \frac{\Sigma_i}{\sigma_i} \right) X^\frac{\beta - 1}{\gamma} Y^\beta, \quad i = 1, 2. \]

We see that if the income correlation coefficient \( \Sigma_i \) is small enough, more specifically, \( \Sigma_i < \frac{\mu_i - r}{\gamma \sigma_i^2} \), then, given \((W,Y)\) the asset allocation in asset \( i \), \( z_i \) for the constrained investor is lower than the allocation for the unconstrained investor.\(^{15}\)

We can also show that, in the non-binding region, the asset allocations in the risky assets \( z_i, i = 1, 2 \), are increasing as functions of income, provided that \( \frac{\mu_i - r}{\gamma \sigma_i^2} > \frac{\Sigma_i}{\sigma_i} \). Fixing wealth in relationship (10), we have

\[ 0 = \left( -\frac{A}{\gamma} X^{-\frac{1}{\gamma}} - 1 + \frac{\beta - 1}{\gamma} KX^\frac{\beta - 1}{\gamma} Y^\beta \right) \frac{\partial X}{\partial Y} - B + \beta KX^\frac{\beta - 1}{\gamma} Y^\beta - 1, \]

and

\[ \frac{\partial X}{\partial Y} = \frac{\beta KX^\frac{\beta - 1}{\gamma} Y^{\beta - 1} - B}{\frac{A}{\gamma} X^{-\frac{1}{\gamma}} - 1 - \frac{\beta - 1}{\gamma} KX^\frac{\beta - 1}{\gamma} Y^\beta}. \]

\(^{15}\)It is easy to check that the allocations are not binding for \( X^\frac{1}{\gamma} Y < Z^* \).
Note that, for $X^{\frac{1}{2}}Y \leq Z^*$

$$\frac{A}{\gamma}X^{-\frac{1}{\gamma}-1} - \frac{\beta - 1}{\gamma}KX^{-\frac{1}{\gamma}-1}Y^\beta > 0.$$ 

Then, we have

$$\frac{\partial z_i}{\partial Y} = \left(\frac{\mu_i - r}{\gamma\sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right) \left(-\frac{(\beta - 1)B}{A} - \frac{\beta}{\gamma}X^{-\frac{1}{\gamma}-1}\frac{\partial X}{\partial Y}\right)$$

$$= \left(\frac{\mu_i - r}{\gamma\sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right) X^{-\frac{1}{\gamma}-1} \left(AB + (\beta - 1)^2BKZ^\beta - \beta^2AKZ^{\beta-1}\right), i = 1, 2$$

We have that the function $h : [0, Z^*] \to \mathbb{R}$, with

$$h(Z) = AB + (\beta - 1)^2BKZ^\beta - \beta^2AKZ^{\beta-1},$$

is decreasing, since

$$h'(Z) = \beta(\beta - 1)KBZ^{\beta-2}(Z - \frac{\beta A}{(\beta - 1)B}) < 0,$$

for $Z \leq Z^* \leq \frac{\beta Z}{(\beta - 1)}$. In addition, we have $h(0) = B > 0$ and

$$\frac{\partial z_i^*}{\partial Y}(Z^*) = \frac{z_i^*}{Y} - \left(\frac{\Sigma_i}{\sigma_i} - (\beta - 1)\frac{\mu_i - r}{\gamma\sigma_i^2}\right) \frac{W*}{Y}$$

$$= \left(\frac{\mu_i - r}{\gamma\sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right) \frac{W*}{Z^*}, i = 1, 2,$$

given the expressions of $W^*$ and $z_i^*$ below. This implies that $h(Z^*) > 0$, and we obtain that $h$ is a positive function on the interval $[0, Z^*]$. Therefore $z_i$ is increasing with income exactly when $(\mu_i - r)/(\gamma\sigma_i^2) > \Sigma_i/\sigma_i$. This is the same condition as in the unconstrained case. At the boundary $X^{\frac{1}{2}}Y = Z^*$, wealth and income are related to the dual variables by

$$W^* = \frac{A - BZ^* + K(Z^*)^\beta Y}{Z^*}.$$

From equation (9), we obtain

$$W^* = \frac{\sum_{i=1}^{2} \left(\frac{\mu_i - r}{\gamma\sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right)\lambda_i (\beta A - (\beta - 1)BZ^*)}{Z^* \left(1 + (\beta - 1)\sum_{i=1}^{2} \frac{\mu_i - r}{\gamma\sigma_i^2}\lambda_i - \beta \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i}\lambda_i\right)} Y.$$

Consumption and asset allocations are given by

$$c^* = \frac{Y}{Z^*}$$

$$z_i^* = \frac{\mu_i - r}{\gamma\sigma_i^2} W^* + B\left(\frac{\mu_i - r}{\gamma\sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right)Y - \beta K(Z^*)^{\beta-1}Y$$

$$= \left(\frac{\mu_i - r}{\gamma\sigma_i^2} - \frac{\Sigma_i}{\sigma_i}\right) + \left(\frac{\Sigma_i}{\sigma_i} \frac{\mu_j - r}{\gamma\sigma_j^2} - \frac{\Sigma_j}{\sigma_j} \frac{\mu_i - r}{\gamma\sigma_i^2}\right) \lambda_j (\beta A - (\beta - 1)BZ^*)$$

$$= \frac{Z^* \left(1 + (\beta - 1)\sum_{i=1}^{2} \frac{\mu_i - r}{\gamma\sigma_i^2}\lambda_i - \beta \sum_{i=1}^{2} \frac{\Sigma_i}{\sigma_i}\lambda_i\right)}{Y}, i, j = 1, 2, i \neq j.$$

37
This implies
\[ z_i^* = \frac{\mu_i - r}{\gamma \sigma_i} - \frac{\Sigma_j}{\sigma_i} + \left( \frac{\Sigma_j}{\sigma_i} - \frac{\mu_i - r}{\gamma \sigma_j} \right) \lambda_j \sum_{i=1}^{2} \frac{\mu_i - r}{\gamma \sigma_i} - \frac{\Sigma_j}{\sigma_i} , \quad i, j = 1, 2, i \neq j. \]

Case 2: Inside the Binding Region with Both Assets. To perform the analysis inside the binding region, given the piecewise linearity of the margin constraint, it turns out to be more convenient to use the primal value function. Recall that \( W = v \) and the Hamilton-Jacobi-Bellman (HJB) equation is
\[ \theta F = \max_{(c \in C, (x,z) \in Q)} \left( c - \frac{1}{1 - \gamma} \left( c \right)^{1 - \gamma} - \frac{c}{W} f'(v) + f'(v) \right) \]
\[ + \left( r - m + \gamma \left( \sum_{i=1}^{2} \Sigma_i \right) \right) f'(v) + \frac{1}{2} \sum_{i=1}^{2} \Sigma_i v^2 f''(v) \]
\[ + \Sigma_i \Sigma_i z_i F_{12} + \frac{1}{2} \sum_{i=1}^{2} \Sigma_i^2 z_i^2 F_{11} , \]
or equivalently
\[ \left( \theta + (\gamma - 1)(m - \gamma \sum_{i=1}^{2} \frac{\Sigma_i^2}{2}) \right) f(v) = \max_{(c \in C, (x,z) \in Q)} \left( c - \frac{1}{1 - \gamma} \left( c \right)^{1 - \gamma} - \frac{c}{W} f'(v) + f'(v) \right) \]
\[ + \left( r - m + \gamma \sum_{i=1}^{2} \Sigma_i \right) f'(v) + \frac{1}{2} \sum_{i=1}^{2} \Sigma_i^2 v^2 f''(v) \]
\[ + \sum_{i=1}^{2} ((\mu_i - r - \gamma \sigma_i \Sigma_i) v f'(v) - \sigma_i \Sigma_i v^2 f''(v)) \frac{z_i}{W} \]
\[ + \frac{1}{2} \sum_{i=1}^{2} \left( \frac{\sigma_i z_i}{W} \right)^2 v^2 f''(v) . \]

Non-Binding Region. We have already analyzed this region using the dual approach. The first order conditions are
\[ c = Y (f'(v))^{-\frac{1}{2}} \]
\[ \frac{z_i}{W} = \frac{\Sigma_i}{\sigma_i} - \frac{(\mu_i - r - \gamma \sigma_i \Sigma_i) f'(v)}{\sigma_i^2 v f''(v)} , \quad i = 1, 2. \]

Binding Region. The first order conditions are
\[ c = Y (f'(v))^{-\frac{1}{2}} \]
\[ \lambda_j \left( (\mu_j - r - \gamma \sigma_j \Sigma_j) f'(v) + \left( \frac{\sigma_j}{W} \right)^2 - \sigma_j \Sigma_j \right) v f''(v) \right) = \lambda_i \left( (\mu_j - r - \gamma \sigma_j \Sigma_j) f'(v) + \left( \frac{\sigma_j}{W} \right)^2 - \sigma_j \Sigma_j \right) v f''(v) \right) , \]
for \( i, j = 1, 2, i \neq j. \) The constraint is
\[ \sum_{i=1}^{2} \lambda_i \frac{z_i}{W} = 1 , \]
with \((\lambda_i, \lambda_j) \in \{\lambda^+, -\lambda^-\}\). Hence
\[
\frac{z_i}{W} = \frac{\lambda_i \sigma_j^2 + \lambda_j (\lambda_j \sigma_i \Sigma_i - \lambda_i \sigma_j \Sigma_j)}{\sum_{i \neq j}^2 \lambda_i^2 \sigma_j^2} + \frac{\lambda_j (\lambda_j (\mu_j - r - \sigma_j \Sigma_j) - \lambda_j (\mu_i - r - \sigma_i \Sigma_i))}{\sum_{i \neq j}^2 \lambda_i^2 \sigma_j^2} \frac{f'(v)}{v f''(v)},
\]
for \(i, j = 1, 2, i \neq j\). When the agent optimally holds both two assets, and the margin constraint is binding, the HJB equation is given by the following ordinary differential equation
\[
(\theta + (\gamma - 1)(m - \gamma \sum_{i=1}^2 \frac{\Sigma_i^2}{2}))(f(v) = \frac{\gamma}{1 - \gamma} \left( f'(v) \right)^{\frac{\gamma-1}{\gamma}} + f'(v)
\]
\[
+ \left( r - m + \gamma \sum_{i=1}^2 \Sigma_i^2 + \frac{\gamma \prod_{j=1}^2 \sigma_j^2}{\sum_{i \neq j}^2 \sigma_i^2 \lambda_j} \sum_{i \neq j} \left( \frac{\mu_j - r}{\sigma_j} - \frac{\Sigma_i}{\sigma_i} \right) \left( \lambda_i + \lambda_j \sigma_i \Sigma_i - \lambda_j \sigma_j \Sigma_j \right) \right) v f'(v)
\]
\[
+ \frac{1}{2} \left( \sum_{i=1}^2 \Sigma_i^2 + \frac{\prod_{j=1}^2 \sigma_j^2}{\sum_{i \neq j}^2 \lambda_j} \left( 1 - \sum_{i=1}^2 \frac{\Sigma_i}{\sigma_i} \right)^2 - \left( \sum_{i=1}^2 \frac{\Sigma_i}{\sigma_i} \lambda_i \right)^2 - \left( \sum_{i \neq j} \frac{\Sigma_i}{\sigma_i} \lambda_i \right)^2 \right) v^2 f''(v)
\]
\[
- \frac{1}{2} \sum_{i, j}^2 \left( \lambda_i^2 \sigma_j^2 \left( \frac{\mu_j - r}{\sigma_j} - \frac{\Sigma_i}{\sigma_i} \right) - \lambda_j^2 \sigma_i^2 \left( \frac{\mu_i - r}{\sigma_i} - \frac{\Sigma_j}{\sigma_j} \right) \right)^2 \left( f'(v) \right)^2 \frac{f''(v)}{v f''(v)}.
\]

**Case 3: Inside the Binding Region with a Single Asset.** The cut off point \(y_j^*\) where the agent optimally chooses to only hold asset \(j\) is such that \(z_i = 0\) and therefore
\[
y_j^* = \frac{\lambda_j (\lambda_i (\mu_j - r - \sigma_j \Sigma_j) - \lambda_j (\mu_i - r - \sigma_i \Sigma_i))}{\lambda_i \sigma_j^2 + \lambda_j (\lambda_j \sigma_i \Sigma_i - \lambda_i \sigma_j \Sigma_j)},
\]
and the condition for holding only asset \(j\) versus holding only asset \(i\) is \(y_j^* > y_i^*\). When the agent optimally only holds asset \(j\), the HJB satisfies the following ODE
\[
\left( \theta + (\gamma - 1)(m - \gamma \sum_{i=1}^2 \frac{\Sigma_i^2}{2} \right) f(v) = \frac{\gamma}{1 - \gamma} \left( f'(v) \right)^{\frac{\gamma-1}{\gamma}} + f'(v)
\]
\[
+ \left( r - m + \gamma \sum_{i=1}^2 \Sigma_i^2 + \frac{\mu_j - r - \sigma_j \Sigma_j}{\lambda_j} \right) v f'(v)
\]
\[
+ \frac{1}{2} \left( \left( \frac{\sigma_j}{\lambda_j} - \Sigma_j \right)^2 + \Sigma_i^2 \right) v^2 f''(v).
\]
The proof of Proposition 8 shows that the boundary condition at \(\frac{W}{Y} = 0\), is given by
\[
f(0) = \frac{1}{(1 - \gamma) \left( \theta + (\gamma - 1)(m - \gamma \sum_{i=1}^2 \frac{\Sigma_i^2}{2} \right)}.
\]

**Proof of Proposition 7.** To show that consumption increases with current wealth and income, we note that
\[
W = -J_1(X, Y),
\]

39
and since \( c = X^{-\frac{1}{\gamma}} \) we obtain
\[
W = -J_1(c^{-\gamma}, Y),
\]
which implies that
\[
\frac{\partial c}{\partial W} = \frac{c^{1+\gamma}}{\gamma J_{11}(X,Y)} > 0,
\]
since \( J \) is strictly convex in \( X \). We also have
\[
\frac{\partial c}{\partial Y} = c^{1+\gamma} J_{12}(X,Y) \gamma J_{11}(X,Y),
\]
It follows that consumption increases with income exactly when \( J_{12} \) is positive. For \( X^{\frac{1}{\gamma}} Y \leq Z^* \), we have
\[
J_{12}(X,Y) = K X^{\frac{2-1}{\gamma}} Y^{\beta-1} > 0.
\]
Then, for \( X^{\frac{1}{\gamma}} Y > Z^* \), assume that at some point \((X_0,Y_0)\) we have \( J_{12}(X_0,Y_0) = 0 \). Since
\[
\gamma X J_{11}(X,Y) = -J_1(X,Y) + Y J_{12}(X,Y),
\]
this implies that
\[
\gamma X_0 J_{11}(X_0,Y_0) = -J_1(X_0,Y_0),
\]
and therefore
\[
a^*(X_0,Y_0) = \frac{\gamma \prod_{i=1}^{2} \sigma_i^2 \left( \sum_{i=1}^{2} \frac{\mu_i - r}{\gamma \sigma_i^2} \lambda_i - 1 \right)}{\sum_{i \neq j} \sigma_i^2 \lambda_j} < 0,
\]
which is impossible. It follows that for all \((X,Y) \in \mathbb{R}_+^2\), \( J_{12} \) must have a constant sign, i.e., \( J_{12} \) is positive. As a consequence, consumption is always increasing with income, so
\[
\frac{\partial c}{\partial Y} > 0. \quad \blacksquare
\]

**Proof of Proposition 8.** The consumption at the limit where the ratio of current wealth to income tends to zero, can be derived by an asymptotic expansion around \( v = 0 \). We postulate that
\[
f(v) \sim d_0 + v - d_1 v^{\frac{3}{2}} + d_2 v^2 + o(v^2),
\]
for some constants \( d_0, d_1 > 0 \) and \( d_2 \) to be determined. This implies that
\[
f'(v) \sim 1 - \frac{3}{2} d_1 v^{\frac{1}{2}} + 2d_2 v + o(v)
\]
\[
f''(v) \sim \frac{3d_1}{4v^{\frac{1}{2}}} + 2d_2 + o(1).
\]
Hence

\[ d_0 = \frac{1}{(1 - \gamma) \left( \theta + (\gamma - 1)(m - \gamma \sum_{i=1}^{2} \frac{\Sigma_i^2}{2}) \right)}. \]

and

\[ \theta + (\gamma - 1)(m - \gamma \sum_{i=1}^{2} \frac{\Sigma_i^2}{2}) = \frac{9}{8\gamma} d_1^2 + (r - m + \gamma \sum_{i=1}^{2} \Sigma_i^2 + \frac{\mu_j - r - \gamma \sigma_j \Sigma_j}{\lambda_j}). \]

It follows that

\[ d_1 = \frac{2 \sqrt{2\gamma (\theta + \gamma (m - (\gamma + 1) \sum_{i=1}^{2} \frac{\Sigma_i^2}{2})) - (r + (\mu_j - r - \gamma \sigma_j \Sigma_j)/\lambda_j)}}{3} > 0. \]

This implies that

\[ c \sim Y. \]
6 References


16. W. Goetzmann and A. Kumar, Why Do Individual Investors Hold Under-Diversified Portfolios?, *Working Paper, Department of Finance, Yale School of Management, 2005*


Figure 1. Optimal asset allocation for two period model

Two assets / Non-binding region
\[ c_0 = 0.5(W+Y), z_1 = 0.13c_0, z_2 = 0.31c_0, x > 0 \]

Two assets / Binding region
\[ z_1 + z_2 = W - c_0, x = 0 \]

One asset / Binding region
\[ z_2 = 0, x = 0 \]

No assets / Binding region
\[ c_0 = W, z_1 = z_2 = x = 0 \]
### Table I
Parameter Values for the Base Case

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Risk aversion $\gamma$</td>
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</tr>
<tr>
<td>Long margin $\lambda^+$</td>
<td>0.5</td>
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<tr>
<td>Short margin $\lambda^-$</td>
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<tr>
<td>Interest rate $r$</td>
<td>6% per year</td>
</tr>
<tr>
<td>Subjective discount rate $\theta$</td>
<td>2% per year</td>
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<tr>
<td>Asset 1 drift $\mu_1$</td>
<td>9% per year</td>
</tr>
<tr>
<td>Asset 1 volatility $\sigma_1$</td>
<td>15% per square root of year</td>
</tr>
<tr>
<td>Asset 2 drift $\mu_2$</td>
<td>12% per year</td>
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<tr>
<td>Asset 2 volatility $\sigma_2$</td>
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<tr>
<td>Income growth rate $m$</td>
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<tr>
<td>Income volatility with respect to first factor $\Sigma_1$</td>
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<tr>
<td>Income volatility with respect to second factor $\Sigma_2$</td>
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Table II: Asset Allocation and Diversification Measures for Basecase Parameters

This table presents the optimal asset allocations and diversification measures for the parameters listed in Table I. WTEWR is the current Wealth-To-Effective-Wealth-Ratio, Ass1 is allocation in asset 1 as a percentage of current wealth, Ass2 is the allocation in asset 2 as a percentage of current wealth, \( \mu_h \) is the expected excess return of the risky part of the portfolio, \( \sigma_h \) is the standard deviation of the excess return of the risky part of the portfolio, \( \sigma_{i,h} \) is the idiosyncratic standard deviation of the excess return, IVarS is the idiosyncratic variance share of the excess returns, \( S_h \) is the Sharpe ratio of the portfolio, RSRL\(_h\) is the relative Sharpe ratio loss, RL\(_h\) is the relative loss of the total portfolio, UL\(_h\) is the utility loss, and \( \beta_h \) is the \( \beta \) of the portfolio with respect to the market portfolio in a market without financial constraints.

<table>
<thead>
<tr>
<th>WTEWR (%)</th>
<th>Ass1 (%)</th>
<th>Ass2 (%)</th>
<th>( \mu_h ) (%)</th>
<th>( \sigma_h ) (%)</th>
<th>( \sigma_{i,h} ) (%)</th>
<th>IVarS (%)</th>
<th>( S_h ) (%)</th>
<th>RSRL(_h) (%)</th>
<th>RL(_h) (%)</th>
<th>UL(_h) (%)</th>
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Table III: Comparative Statics: Sharpe Ratio
This table presents the optimal asset allocations and diversification measures for the case where all the parameters are given in in Table I, other than the volatility of the second asset, which is set to 40%. WTEWR is the current Wealth-To-Effective-Wealth-Ratio. Ass1 is allocation in asset 1 as a percentage of current wealth. Ass2 is the allocation in asset 2 as a percentage of current wealth. $\mu_h$ is the expected excess return of the risky part of the portfolio, $\sigma_h$ is the standard deviation of the excess return of the risky part of the portfolio, $\sigma_{i,h}$ is the idiosyncratic standard deviation of the excess return, IVarS is the idiosyncratic variance share of the excess returns, $S_h$ is the Sharpe ratio of the portfolio, $RSRL_h$ is the relative Sharpe ratio loss, $RL_h$ is the relative loss of the total portfolio, $UL_h$ is the utility loss, and $\beta_h$ is the $\beta$ of the portfolio with respect to the market portfolio in a market without financial constraints.

<table>
<thead>
<tr>
<th>WTEWR (%)</th>
<th>Ass1 (%)</th>
<th>Ass2 (%)</th>
<th>$\mu_h$ (%)</th>
<th>$\sigma_h$ (%)</th>
<th>$\sigma_{i,h}$ (%)</th>
<th>IVarS (%)</th>
<th>$S_h$ (%)</th>
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Table IV: Comparative Statics: Margin Requirements
This table presents the optimal asset allocations and diversification measures for the case where all the parameters are given in Table I, other than the margin requirement for risky assets, which is set to 100%; i.e., an investor can not short a risky asset. WTEWR is the current Wealth-To-Effective-Wealth-Ratio, Ass1 is allocation in asset 1 as a percentage of current wealth, Ass2 is the allocation in asset 2 as a percentage of current wealth, $\mu_h$ is the expected excess return of the risky part of the portfolio, $\sigma_h$ is the standard deviation of the excess return of the risky part of the portfolio, $\sigma_{i,h}$ is the idiosyncratic standard deviation of the excess return, $\text{IVarS}$ is the idiosyncratic variance share of the excess returns, $S_h$ is the Sharpe ratio of the portfolio, $\text{RSRL}_h$ is the relative Sharpe ratio loss, $RL_h$ is the relative loss of the total portfolio, $UL_h$ is the utility loss, and $\beta_h$ is the $\beta$ of the portfolio with respect to the market portfolio in a market without financial constraints.

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