Optimal Consumption and Investment Strategies under Wealth Ratcheting

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Abstract

Individuals driven by capital accumulation may be reluctant to experience large wealth downfalls. Implications for optimal consumption and investment policies are explored in a dynamic setting where wealth is restrained from falling below a fraction of its all-time high. Risky investment regulates wealth growth and mitigates the ratchet effect of the constraint, and may decrease as wealth approaches its maximum. The correspondence found between habit formation over consumption and wealth ratcheting provides a rational explanation for the extensive use of such a practice in investment management. An extension embeds the spirit of capitalism using wealth as an index for social status.

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1 INTRODUCTION

The desire for wealth accumulation is well established in the literature. According to Max Weber (1958), "man is dominated by the making of money, by acquisition as the ultimate purpose of his life. Economic acquisition is no longer subordinated to man as the means for the satisfaction of his material needs". The recent explosion and success of capital guarantee funds suggest that investors are looking for downside protection but at the same time upside potential. Fund trusts and institutions such as a university or a foundation may also seek asset preservation. When a non-profit organization receives an endowment and other long-term funding, it has to manage these resources prudently by establishing a spending policy that accommodates the need for asset protection and portfolio growth. Usually donors require endowment assets to be kept permanently and prohibit grantees from using or borrowing against principals. Returns can be used for contributions, or to increase the endowment assets. The aim of such spending rules is to preserve financial independence and to avoid the purchasing power erosion over time[1]. Fund performance is often measured by all-time record levels that seem to be appealing to people, and high-water marks[2] are common in the investment management industry. For instance, some financial services firms offer their customers the following portfolio insurance strategy: an investor who stays invested until the fund matures is guaranteed to receive a value equal to the highest value of the fund ever achieved, even if the fund's daily value has fallen since its highest point.

In this paper, we analyze the intertemporal investment-consumption rules for an infinite lived individual maximizing her expected discounted utility under wealth ratcheting. Namely, the agent does not tolerate losing more than a fixed percentage of her all-time high level of wealth. This constraint was first introduced by Grossman and Zhou (1993) who argue that a large drawdown (typically above 25 percent) is often a reason for firing fund managers[3].

The key intuition behind most of the results is driven by two effects. First, as in any portfolio

[1] In the US, trustees and charity professionals who run foundations after a founder's death are only obliged to spend as little as 5% a year of the capital. In many foundations, capricious and poorly thought out projects or programs were undertaken to fulfill the interests of trustee managers not the wishes of the founder (The Economist, May 28th 2005).

[2] The high-water mark is a target value that can depend on the current asset value of the fund, and it is adjusted due withdraws, allocated expenses and a contractual growth rate. In the simplest case, the high-water mark is the highest level the asset has reached in the past.

[3] Grossman and Zhou's (1993) examine the problem of maximizing the long term growth rate of expected utility of final wealth. Their analysis is quite insightful but they do not allow for endogenous withdraws from the fund to finance intermediate consumption. Cvitanic and Karatzas (1995) extend their work to a more general class of stochastic processes by developing a martingale approach.

selection problem under market restrictions, the agent is concerned with hedging motives that in the future the constraint may be binding. As a benchmark, hedging concerns are addressed in a simpler framework when the investor is required to maintain her wealth above a *fixed* floor (foundation charter requirement). Essentially, risk aversion is enhanced, which leads to smaller stock holdings and lower consumption plans with respect to the unconstrained case. Both optimal allocations are found increasing in wealth. Second, the drawdown constraint displays a ratcheting feature since each time financial wealth reaches a new record high, the minimum floor rises and the restriction becomes more stringent. The agent has two margins of adjustment at her disposal to regulate the growth of her wealth: consumption and risky investment. The latter is the most sensitive of the two as it governs the diffusion component of the wealth process. The optimal solution of the model reflects the trade-off between consuming today and deferring consumption to take advantage of investing in the stock market, which may be thwarted by the presence of the ratchet. We derive conditions under which, as wealth approaches its all-time high, the fraction of wealth invested in stocks decreases and possibly is set to zero. In this last case, the maximum to date level of wealth is an upper reflecting (absorbing) barrier if the individual is fairly patient (impatient) with a large (small) intertemporal elasticity of substitution (IES).

Tracking wealth movements, the optimal consumption policy exhibits a ratcheting behavior and large drawdowns from its all-time consumption level are prohibited. We emphasize the correspondence between wealth ratcheting and habit formation in the spirit of Duesenberry (1949). This twin ratcheting is an important result that rationalizes the loss aversion for wealth, in particular for an investor who delegates the management of her wealth and aims at maintaining her standard of living. An extension of the basic model embeds the spirit of capitalism by including wealth, an index of social status, inside the utility function[4]. Persistent benefits derived from building up status lead to a more aggressive risky investment policy whereas consumption becomes less appealing.

This paper builds on the dynamic portfolio choice literature. Early works on optimal consumption-investment allocations in a frictionless market and no borrowing restrictions include Samuelson (1969) and Merton (1971). Then, attention has been paid on more real world situations where investors face constraints in their portfolio investments[5]. In general, the optimal strategy differs from the

[4]For instance see Baski and Chen (1996) and Smith (2001).

[5]Cvitanic and Karatzas (1992) and Cuoco (1997) develop a general martingale approach to cope with convex contemporaneous constraints on trading strategies which includes the case of incomplete markets and prohibited short sales. Cuoco and Liu (2000) analyze the optimal consumption portfolio choice problem under margin requirements and eval-

unconstrained one as the agent aims at hedging against the constraint (at some cost) since even though the constraint may not be binding, there is a possibility that it does in the future. Recent papers focus on portfolio allocations under wealth performance relative to an exogenous benchmark such as in Browne (2001) or subject to growth objectives required by the decision maker as in Hellwig (2003). In Carpenter (2000), the fund manager is compensated with a call option on the wealth she manages with a benchmark index as strike price. The author shows that the option compensation does not necessarily lead to more risk seeking. Goetzmann, Ingersoll and Ross (2003) study hedge fund compensation schemes when managers perceive a regular fee proportional to the portfolio asset value and an incentive fee based on the fund return each year in excess of the high-water mark. Consistent with empirical evidence, they obtain that a significant proportion of managers compensation can be attributed to the incentive fee, in particular for high volatility asset funds for which high manager skills are required.

The paper is also related to the trend of research that strives to provide some alternative to the usual time separable von Neumann-Morgenstein preferences whose performance has been poor from an empirical point of view. In particular, such preferences have failed to explain the equity premium puzzle, i.e. the fact that returns on the stock market exceed on average the return of Treasury bills by an average of six percentage points. Habit formation preferences such as Sundaresan (1989), Constantidines (1990), Detemple and Zapatero (1991) postulate that agents not only derive utility from current consumption but also from consumption history, typically captured by a standard of living index. However, for tractability reasons, many models assume that the agent derives utility from the excess between current consumption and the habit level. If the marginal utility at zero is infinite, the standard of living index acts as a floor level below which current consumption does not fall. This addictive feature - optimal consumption levels can only increase across time regardless of the state of the economy- is not supported by empirical evidence. Detemple and Karatzas (2003) address this issue and investigate the case of finite marginal utility of consumption at zero when imposing a non-negativity constraint on consumption plans. When the shadow price of consumption is high, the agent optimally reduces her consumption along with her standard of living and the associated "cost" of habits as well. An alternative approach proposed by Dybvig (1995) is to ratchet current consumption. Originally, Duesenberry (1949) emphasized that consumption may not be entirely reversible over time

uate the cost of the constraint. He and Pages (1993) and El Karaoui and JeanBlanc-Picqué (1998) treat the case of non-negative wealth in presence of labor income. Grossman and Villa (1992) followed by Villa and Zariphopoulou (1997) study the consumption-portfolio problem for a CRRA investor facing a leverage constraint.

but instead may increase along with income and decline less than proportionally with it. Dybvig (1995) formalizes this idea by looking at an extreme form of habit formation where consumption is prevented from falling over time. With little work, it is possible to extend Dybvig's analysis and assume that the agent is intolerant to any decline that exceeds a fixed proportion of her all-time consumption. In some sense, the model derived here is a mirror problem as we show that imposing ratcheting on wealth induces a ratcheting behavior on consumption with a strong parallel with Dybvig (1995).

Finally, our model can be seen as an example of extreme loss aversion in wealth since utility can be defined to be minus infinity if the drawdown constraint is violated. The concept of loss aversion was first proposed by Kahneman and Tversky (1979 and 1991) and postulates that the impact of a loss is greater than that of an equally sized gain. Barberis, Huang and Santos (2001) explore the implications on asset prices of loss aversion by considering an investor who derives utility not only from consumption but also from changes in the value of her financial wealth. Their model is flexible enough to allow the degree of loss aversion to be affected by prior investment performance.

The paper is organized as follows. Section 2 describes the economic setting and contains the derivation and the analysis of the optimal consumption and portfolio allocations. In section 3, we assess the cost of the drawdown constraint. Section 4 presents an extension of the basic model that embeds the spirit of capitalism using wealth as a proxy for social status. Section 5 concludes. Proofs of all results are collected in the Appendix.

2 THE ECONOMIC SETTING

Time is continuous. An infinitely lived investor, who is reluctant to let her wealth fall more than a fraction of its historical maximum, has to optimally allocate her wealth between a risk-free bond, a risky asset and consumption.

Individual preferences. There is a single perishable good available for consumption, the numéraire. Preferences are represented by a time additive utility function

$$U(c) = E\left[\int_0^\infty u(c_s)e^{-\theta s}ds\right],$$

where the instantaneous utility function u is twice continuously differentiable, increasing and strictly concave and θ denotes the subjective time discount rate. In addition, u satisfies the following Inada conditions: $\lim_{c\to 0^+} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. In the sequel, we focus our analysis on an individual with constant relative risk aversion preferences

$$u(c) = \begin{cases} \frac{c^{1-b}}{1-b}, \ b \neq 1 \\ \ln c, \ b = 1. \end{cases}$$

Information structure and financial market. Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a *one* dimensional (standard) Brownian motion w. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P is assigned. Let $\mathcal{F}_t$ be the σ-algebra generated by the observations of w $\{w_s; 0 \leq s \leq t\}$ and augmented. At time t, the investor's information set is $\mathcal{F}_t$. The filtration $\mathbb{F} = \{\mathcal{F}_t, \ t \in \mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented). All the processes considered in the sequel are progressively measurable with respect to $\mathbb{F}$ and all identities involving random variables (respectively stochastic processes) should be understood to hold $P - a.s.$ (respectively, $(Lb \times P) - a.e.$, where Lb denotes the Lebesgue measure on $\mathbb{R}_+$).

There are two securities available in the financial market:

- a risk-free bond whose price B evolves according to

$$dB_s = rB_s ds,$$

where r is the constant interest rate, and,

- an index modeled by a risky security whose price S follows a geometric Brownian motion

$$dS_s = S_s \left(\mu ds + \sigma dw_s\right),$$

where dw_s is the increment of a standard Wiener process, μ is the mean return of the stock index S and σ^2 is its instantaneous variance. Let x and z be respectively the *amount* of dollars invested in the riskless bond B and risky security S, so that the wealth process W is equal to $x + z$. A consumption plan c is feasible if there is a trading strategy (x, z) such that

$$\begin{aligned} dW_s &= (rW_s - c_s + z_s(\mu - r))ds + \sigma z_s dw_s, \\ W_s &> -\underline{K}, \end{aligned} \tag{1}$$

with $\underline{K} > 0$. The condition $W_s > -\underline{K}$ rules out arbitrage opportunities, such as doubling strategies presented in Harrison and Kreps (1979). Finally, trading strategies (x, z) and consumption plans c are adapted processes satisfying the standard integrability conditions

$$\int_0^\infty c_s^2 ds < \infty, \ \int_0^\infty |rx_s| \, ds + \int_0^\infty |\mu z_s| \, ds + \int_0^\infty \sigma^2 z_s^2 ds < \infty.$$

Drawdown constraint. Let $M_t = \sup_{0 \le s \le t} \{W_s, M_0\}$ be the maximum to date t level of wealth. As introduced in Grossman and Zhou (1993), the drawdown constraint is

$$W_s \ge \alpha M_s, \tag{2}$$

for some α in $[0,1]$. This constraint indicates that the investor is reluctant to let her wealth fall below a fraction of its maximum to date. In the investment management industry, a realistic estimate of α ranges from 75 to 88 percent. In practice, different values of α may apply to different types of traders. For instance, for proprietary traders (internal hedge fund traders) who invest money belonging to their company, α can depend on the target amount of money a trader is required to generate during the year and could be as high as 94 percent.

We first review the main results for the unconstrained problem studied by Merton (1971).

2.1 Benchmark case: Merton problem

Within our financial market framework, the Merton problem (1971) for a CRRA investor is

$$F(W_t) = \max_{(c,z)} E_t \left[\int_t^\infty \frac{c_s^{1-b}}{1-b} e^{-\theta(s-t)} ds \right],$$

subject to the budget constraint (1) and $W_t > 0$ given. The transversality condition for this problem is

$$\lim_{T \to \infty} E_t \left[F(W_{t+T}) e^{-\theta(t+T)} \right] = 0.$$

Merton (1971) shows that both the fraction of wealth invested in stocks $\frac{z_s^f}{W_s}$ and the consumption-wealth ratio $\frac{c_s^f}{W_s}$ are constant and given by

$$\begin{aligned} \frac{z_s^f}{W_s} &= \frac{\mu - r}{b\sigma^2} \\ \frac{c_s^f}{W_s} &= \frac{1}{A}, \end{aligned}$$

where $A^{-1} = \frac{\theta}{b} + \frac{b-1}{b}\left(r + \frac{(\mu-r)^2}{2b\sigma^2}\right) > 0$. The (optimal) wealth process W^f is a geometric Brownian motion whose dynamics are

$$dW_t^f = W_t^f \left((r - \frac{1}{A} + \frac{(\mu - r)^2}{b\sigma^2}) dt + \frac{\mu - r}{b\sigma} dw_t \right).$$

In order to gain insights about the effects of the drawdown constraint (2), we examine the simpler consumption-portfolio choice problem where wealth is required to be kept above a *fixed* minimum floor adjusted for inflation. In particular, this allows us to isolate and quantify hedging motives.

2.2 Fixed minimum floor problem

Consider a foundation whose charter stipulates that the endowment $\alpha\underline{M} > 0$ adjusted for inflation with rate $\lambda > 0$ cannot be used for expenditures (only the returns are eligible). No other constraint is assumed regarding the growth objectives of the trust fund of the foundation. At any time t, wealth W_t must be maintained above a minimum level $\alpha\underline{M}e^{\lambda t}$. Let us define $\widehat{W}_t \equiv W_t e^{-\lambda t}$, $\widehat{c}_t \equiv c_t e^{-\lambda t}$ and $\widehat{z}_t \equiv z_t e^{-\lambda t}$. Given the linearity of the wealth dynamics and the homogeneity of the utility function, the investor's problem can be written

$$
\begin{aligned}
F(\widehat{W}_t) &= \max_{(\widehat{c},\widehat{z})} E_t\left[\int_t^\infty \frac{\widehat{c}_s^{\,1-b}}{1-b} e^{-\theta'(s-t)} ds\right] \\
\text{s.t. } d\widehat{W}_s &= \left(r'\widehat{W}_s - \widehat{c}_s + \widehat{z}_s(\mu' - r')\right) ds + \sigma \widehat{z}_s dw_s \\
&\widehat{W}_s \geq \alpha\underline{M},\ \widehat{W}_t > 0 \text{ given,}
\end{aligned}
\tag{P}
$$

where the parameters adjusted for inflation are $r' = r - \lambda$, $\mu' = \mu - \lambda$, and the adjusted time discount rate $\theta' = \theta + (b-1)\lambda$ is assumed to be positive. The transversality condition is the same as before. We still require $A > 0$ and in addition we make the following assumptions:

A1. The interest rate r' is positive.

A2. The Sharpe ratio of the risky asset is positive.

Assumption A1. is required for feasibility. Assumption A2. is made for convenience and without loss of generality.

First of all, note that the value function F is increasing and concave[6] in $\widehat{W}$. Then, for $\widehat{W} \geq \alpha\underline{M}$, the Hamilton Jacobi Bellman (HJB) equation of this problem is

$$
\theta' F = \max_{(\widehat{c},\widehat{z})} \frac{\widehat{c}^{\,1-b}}{1-b} + \left(r'\widehat{W} - \widehat{c} + \widehat{z}(\mu' - r')\right) F' + \frac{\sigma^2}{2}(\widehat{z})^2 F''. \tag{3}
$$

The optimal conditions are

$$
\begin{aligned}
\widehat{c}^* &= (F')^{-\frac{1}{b}} \\
\widehat{z}^* &= -\frac{(\mu' - r')F'}{\sigma^2 F''},
\end{aligned}
$$

and F satisfies the following non-linear ODE

$$
\theta' F = \frac{b(F')^{\frac{b-1}{b}}}{1-b} + r'\widehat{W}F' - \frac{1}{2}\left(\frac{\mu' - r'}{\sigma}\right)^2 \frac{(F')^2}{F''}. \tag{4}
$$

[6]The strict concavity of F comes from the fact that the utility function is strictly concave and the constraint is linear so that if W and $W'$are admissible wealth processes, then for all λ in $[0,1]$, $\lambda W + (1-\lambda)W'$ is also admissible.

Lemma 1 *The general solution of ODE (4) is such that*

$$\widehat{W} = A(F'(\widehat{W}))^{-\frac{1}{b}} + L_1(F'(\widehat{W}))^{\frac{\beta_1'-1}{b}} + L_2(F'(\widehat{W}))^{\frac{\beta_2'-1}{b}}, \tag{5}$$

where β_1' *and* β_2' *are respectively the positive and negative roots of the quadratic*

$$\frac{1}{2}\left(\frac{\mu'-r'}{b\sigma}\right)^2 x^2 + \left(\frac{1}{A} - r' - \frac{1}{2}\left(\frac{\mu'-r'}{b\sigma}\right)^2\right) x = \frac{1}{A},$$

and L_1 *and* L_2 *are two constants to be determined.*

Proof. See the Appendix. ■

Useful results $\beta_1' > 1$ and $1 - b - \beta_2' > 0$ are proved in the Appendix.

Boundary Condition at the Minimum Floor. At $\widehat{W} = \alpha\underline{M}$, we have

$$\alpha\underline{M} = A(F'(\alpha\underline{M}))^{-\frac{1}{b}} + L_1(F'(\alpha\underline{M}))^{\frac{\beta_1'-1}{b}} + L_2(F'(\alpha\underline{M}))^{\frac{\beta_2'-1}{b}},$$

and in order not violate the constraint with some positive probability in a near future, stock holdings must be zero, which implies

$$A = (\beta_1' - 1)L_1(F'(\alpha\underline{M}))^{\frac{\beta_1'}{b}} + (\beta_2' - 1)L_2(F'(\alpha\underline{M}))^{\frac{\beta_2'}{b}}.$$

When $\widehat{W}$ is large, the constraint is equivalent to $\widehat{W} \geq 0$, so the solution is equivalent to the one for the unconstrained case, i.e. $F'(\widehat{W}) \underset{\infty}{\sim} A^b(\widehat{W})^{-b}$. Since $\frac{\beta_2'-1}{b} < -\frac{1}{b}$, we must have $L_2 = 0$. At $\widehat{W} = \alpha\underline{M}$, the consumption-wealth ratio and the constant L_1 are given by

$$\begin{aligned} \frac{c}{\alpha\underline{M}} &= \frac{\beta_1'-1}{\beta_1' A} < \frac{1}{A} \\ L_1 &= \left(\frac{\alpha\underline{M}}{\beta_1'}\right)^{\beta_1'}\left(\frac{A}{\beta_1'-1}\right)^{1-\beta_1'} > 0. \end{aligned}$$

Note that at $\widehat{W} = \alpha\underline{M}$, the wealth dynamics are deterministic

$$d\widehat{W}_t = \left(r' - \frac{\beta_1'-1}{\beta_1' A}\right)\widehat{W}_t dt.$$

It is easy to see that $r' - \frac{\beta_1'-1}{\beta_1' A} = \frac{1}{2}\left(\frac{\mu'-r'}{b\sigma}\right)^2(\beta_1' - 1)$ is positive, which means that wealth bounces back upward after hitting the minimum floor[7].

[7]This property is actually necessary for a well defined problem. In the sequel, when the drawdown constraint (2) is imposed, restrictions on the parameters of the model are made so that this "reflecting condition" is satisfied.

2.2.1 Properties of the optimal allocations

The consumption-wealth ratio $\frac{\widehat{c}^*}{W}$ is given by

$$\frac{\widehat{c}^*}{W} = \frac{1}{A + L_1(F'(W))^{\frac{\beta_1'}{b}}}.$$

It is increasing in wealth and smaller than in the unconstrained case. The fraction of wealth invested in the stock is given by

$$\frac{\widehat{z}^*}{W} = \frac{\mu - r}{b\sigma^2}\left(1 - \beta_1' + \frac{\beta_1' A}{A + L_1(F'(W))^{\frac{\beta_1'}{b}}}\right).$$

This ratio is monotonic (increasing) in wealth and smaller with respect to the unconstrained case. The reason is the rise of the relative risk aversion of the lifetime utility in wealth since

$$-\frac{WF''}{F'} = b\left(1 + \frac{\beta_1' L_1(F'(W))^{\frac{\beta_1'}{b}}}{A + (1-\beta_1')L_1(F'(W))^{\frac{\beta_1'}{b}}}\right) > b.$$

At the floor $W = \alpha M$, this relative risk aversion is infinite and consequently holdings in stock are zero. Note that the risky investment strategy is not of CPPI (that is, constant proportion portfolio insurance) type as proposed by Black and Perold (1992) and optimally derived by Grossman and Zhou (1993) for a stochastic floor. As wealth increases, lifetime utility relative risk aversion decreases and as wealth becomes very large, the effects of the constraint vanish: optimal allocations converge to the optimal unconstrained ones.

Our analysis so far has shown that in presence of a *fixed* minimum floor, hedging motives induce a reduction in consumption and risky investment and enhance risk aversion. In the next section, we will see that the ability of the individual to control the minimum floor combined with a ratchet effect lead to quite different properties of stock holdings as well as for consumption plans as they serve as wealth growth regulators.

2.3 Consumption-portfolio choice problem with a drawdown constraint

The agent aims at maximizing her lifetime utility

$$F(W_t, M_t) = \max_{(c,z)} E_t\left[\int_t^\infty \frac{c_s^{1-b}}{1-b} e^{-\theta(s-t)} ds\right],$$

subject to constraints (1) and (2), with $W_t > 0$, $M_t > 0$ given.

Transversality Condition. The transversality condition for this problem is:

$$\lim_{T\to\infty} E_t\left[F(W_{t+T}, M_{t+T})e^{-\theta(t+T)}\right] = 0.$$

As before, we assume that A and r are positive as well as a positive Sharpe ratio. Further assumptions on the parameters are made in the sequel to ensure feasibility. We start the analysis by reviewing some useful properties of the maximum process M and the value function F.

2.3.1 Properties of the maximum process

P1. As mentioned in Grossman and Zhou (1993), M is a continuous increasing process and thus a finite variation process.

P2. Denoting by $[X, Y]$ the quadratic covariation between processes X and Y, we have $d\,[M, W]_t = 0$ and $d\,[M, M]_t = 0$.

2.3.2 Properties of the value function

P1. F is strictly increasing and concave in W and decreasing in M.

P2. F is homogenous of degree $1 - b$ in (W, M).

Proof. See the Appendix. ■

Property **P2** implies that

$$F(W, M) = M^{1-b} f(u),$$

with $u = \frac{W}{M}$ and some smooth function f. Note that from property **P1** f is also concave and strictly increasing in u.

2.3.3 Derivation of the value function.

Given the properties of the maximum process M, for $W \in (\alpha M, M)$, the HJB associated to the investor's program is

$$\theta F = \max_{(c,z)} \frac{c^{1-b}}{1-b} + (rW - c + z(\mu - r))\, F_1 + \frac{\sigma^2}{2} z^2 F_{11}. \tag{6}$$

The optimal conditions can be rewritten

$$\begin{aligned} c^* &= M(f'(u))^{-\frac{1}{b}} \\ \frac{z^*}{W} &= -\frac{(\mu - r)f'(u)}{\sigma^2 u f''(u)}, \end{aligned}$$

and for $u \in (\alpha, 1)$ the function f satisfies the non-linear ODE

$$\theta f(u) = \frac{b(f'(u))^{\frac{b-1}{b}}}{1-b} + ruf'(u) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(f'(u))^2}{f''(u)}. \tag{7}$$

As shown in lemma 1, the general solution f of the ODE (7) is such that

$$u = A(f'(u))^{-\frac{1}{b}} + K_1(f'(u))^{\frac{\beta_1 - 1}{b}} + K_2(f'(u))^{\frac{\beta_2 - 1}{b}}, \tag{8}$$

where β_1 and β_2 are respectively the positive and negative roots of the quadratic

$$\frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2 x^2 + \left(\frac{1}{A} - r - \frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2\right) x = \frac{1}{A}, \tag{9}$$

and K_1 and K_2 are two constants to be determined. In the sequel, we find that $K_1 > 0$ and $K_2 < 0$.

Interpretation of the solution The optimal wealth process is the sum of three terms:

$$W = AM(f'(u))^{-\frac{1}{b}} + K_1 M(f'(u))^{\frac{\beta_1 - 1}{b}} + K_2 M(f'(u))^{\frac{\beta_2 - 1}{b}}.$$

The first one is the usual consumption term as in Merton problem. The second term is positive and incorporates hedging motives as in the fixed minimum floor problem. This term can be related to portfolio insurance strategies involving simple options such as in Black and Perold (1992). Finally, the third term is negative and regulates the growth rate of the wealth to mitigate the ratchet effect of the stochastic floor.

The focus of the next paragraph is to establish the boundary conditions at $u = \alpha$ and $u = 1$.

2.3.4 Boundary conditions

The boundary conditions are derived in the Appendix. To sum up, at $u = \alpha$, as in the minimum floor problem, holdings in the risky asset must be zero. At $u = 1$, the condition must ensure that the Hamilton Jacobi Bellman equation still holds. There are two possibilities depending on the parameters: either $F_2(M, M) = 0$ or holdings in the risky asset is set to zero. Denoting $Y = (f'(1))^{\frac{1}{b}}$ and $X = (f'(\alpha))^{\frac{1}{b}}$, the boundary conditions are

$$\begin{aligned}
\alpha X &= A + K_1 X^{\beta_1} + K_2 X^{\beta_2} \\
A &= (\beta_1 - 1)K_1 X^{\beta_1} + (\beta_2 - 1)K_2 X^{\beta_2} \\
Y &= A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2} \\
A - (\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2} &= \max\left\{0, \frac{1 - b - \beta_1\beta_2}{b - 1}(A_0 - Y)\right\} \quad b \neq 1,
\end{aligned}$$

where $A_0^{-1} = \frac{\theta+(b-1)r}{b}$ and when $b = 1$, $Y = A = A_0 = \frac{1}{\theta}$. The following proposition specifies the optimal holdings in stock when $u = 1$.

Proposition 1 *Whenever $b \geq 1$ ($b \leq 1$), as long as $Y \leq A_0$ ($Y \geq A_0$), the optimal boundary condition at $W = M$ is $F_2(M, M) = 0$ and stock holdings are positive, $z_1^* > 0$. Otherwise, setting the risky portfolio allocation to zero, $z_1^* = 0$, is the optimal boundary condition at $W = M$.*

Proof. See the Appendix. ■

As developed in more details in the sequel, the intuition behind the results of proposition 1. is the agent's willingness of mitigating the ratchet impact (and the irreversible associated cost) of the drawdown constraint. The existence and uniqueness of the quadruple (K_1, K_2, X, Y) with $K_1 > 0$ and $K_2 < 0$ are shown in the Appendix.

2.3.5 Reflecting condition

As already mentioned in the section 2.2, when the drawdown constraint binds, the wealth dynamics are deterministic

$$dW_t = (r - \frac{1}{X})W_t dt.$$

In order for the wealth process W to remain above the minimum floor αM in the next instant, we must have $r > \frac{1}{X}$.

Having solved the HJB equation and determined the boundary conditions at $u = \alpha$ and $u = 1$, we now analyze the properties of the optimal allocations.

2.4 Properties of the optimal allocations

2.4.1 Consumption

Optimal consumption c^* is implicitly defined by the relationship

$$\frac{W}{M} = G\left(\frac{c^*}{M}\right), \tag{10}$$

where $G(x) = Ax + K_1 x^{1-\beta_1} + K_2 x^{1-\beta_2}$ and since $G' > 0$, it is increasing in current wealth W. The consumption wealth ratio is given by

$$\frac{c^*}{W} = \frac{1}{u}\left(f'(u)\right)^{-\frac{1}{b}},$$

so

$$\frac{\partial}{\partial u}\left(\frac{c^*}{W}\right) = \frac{1}{bu^2}\left(f'(u)\right)^{-\frac{1}{b}}\left(-\frac{uf''(u)}{f'(u)} - b\right) > 0,$$

since due to hedging motives, we establish in the sequel that $\frac{z^*}{W} < \frac{\mu - r}{b\sigma^2}$, which implies that the lifetime utility relative risk aversion $-\frac{uf''(u)}{f'(u)}$ is above its unconstrained level b.

The consumption-wealth ratio $\frac{c^*}{W}$ is increasing in the ratio current wealth over its peak, so in particular increasing in current wealth and decreasing in the historical maximum level of wealth. At the ceiling $W = M$, we have $\frac{c^*}{M} = \frac{1}{Y}$, so in particular, for $b > 1$, $Y > A$ (see the Appendix), we can conclude that for all u in $[\alpha, 1]$, $\frac{c^*}{W} < \frac{1}{A}$. Recall that the intertemporal elasticity of substitution (IES) s is equal to $\frac{1}{b}$. Hence if the investor is reluctant ($s < 1$) to alter her consumption plans overtime, she chooses to consume a lower fraction of her wealth than she does in the unconstrained case. Conversely, when $b < 1$, we have $Y < A$. Therefore, when the investor is willing to alter her consumption plans ($s > 1$), for $\frac{W}{M}$ large enough, the consumption-wealth ratio is larger than in the unconstrained case. For α close to 1, this property is global[8] in the sense that for all u in $[\alpha, 1]$, $\frac{c^*}{W} > \frac{1}{A}$.

Next, we show that optimal consumption inherits a ratcheting behavior from wealth and habit formation endogenously arises.

All-Time High Consumption and Habit Formation. Denoting $c^*_{M_t} = \sup_{0 \le s \le t} \{c^*_s\}$ the maximum to date level of consumption, for $0 \le s \le t$, we have $\frac{1}{X} \le \frac{c^*_s}{M_s} \le \frac{1}{Y}$. Since $M_s \le M_t$, it follows that for all date t,

$$\frac{Y}{X} \le \frac{c^*_t}{c^*_s} \le \frac{c^*_t}{c^*_{M_t}}.$$

The current consumption level c^*_t over its peak $c^*_{M_t}$ remains within the fixed band $[\alpha_c, 1]$, with $\alpha_c = \frac{Y}{X} < 1$. The maximum drawdown in consumption from its previous all-time high is $1 - \alpha_c$ and it decreases as α goes up (see the Appendix). Imposing ratcheting on the wealth process induces a ratcheting behavior of the optimal consumption as posited by Duesenberry (1949) and analytically derived by Dybvig (1995). When the investor does not tolerate any decline in consumption, Dybvig establishes that for all times t, $\frac{c^*_{M_t}}{r} \le W_t \le \frac{-\beta_2 c^*_{M_t}}{r(1-\beta_2)}$. This implies that current wealth W_t must be kept above the proportion $\frac{-\beta_2}{1-\beta_2}$ of its peak M_t. Grossman and Zhou (1993) claim that the reason for such a restriction on the manager's investment policy is that the owner of the fund psychologically (and often physically) commits to use part of the profit when reaching the peak. Dybvig argues that imposing a

[8] In the limit case $\alpha = 1$, we show in the sequel that the consumption-wealth ratio is equal to $\frac{1}{A_0}$ and that $X = Y = A_0$. Since for $b > 1$, $\frac{1}{A_0} > \frac{1}{A}$, by continuity, we deduce that for large values of the drawdown coefficient α, we have $\alpha X < A$ and this implies $\frac{c^*}{W} > \frac{1}{A}$, for all u in $[\alpha, 1]$.

drawdown constraint on wealth seems ad hoc from an economic point of view, and his motivation was to offer an alternative to the work by Grossman and Zhou (1993). Although the problem studied here and Dybvig's model are not equivalent, our analysis provides a bridge between the two approaches as well as an economic justification in terms of preferences (habit formation) over consumption for downside protection on wealth. The drawdown constraint (2) is a practical and effective way to ensure that standard of living will not have to be lowered by too much in the case of an adverse shock.

We now investigate the impact of the magnitude of the drawdown proportion α on the consumption-wealth ratio.

Proposition 2 *If $z_1^* = 0$ is optimal, the more stringent the drawdown constraint (higher α), the smaller the consumption-wealth ratio for all u in $[\alpha, 1]$. When $z_1^* > 0$ is optimal, if $b \geq 1$, the previous result remains valid. However, if $b < 1$, there is a critical value u_α^* in $(\alpha, 1)$, such that the consumption-wealth ratio decreases in α on $[\alpha, u_\alpha^*]$ and increases on $[u_\alpha^*, 1]$.*

Proof. See the Appendix. ■

Proposition 2 suggests that for an investor with a high IES ($s > 1$), when wealth is about to reach its peak, for large values of α, the investor relies on the consumption margin to regulate the growth of her wealth and dampen the ratchet effect.

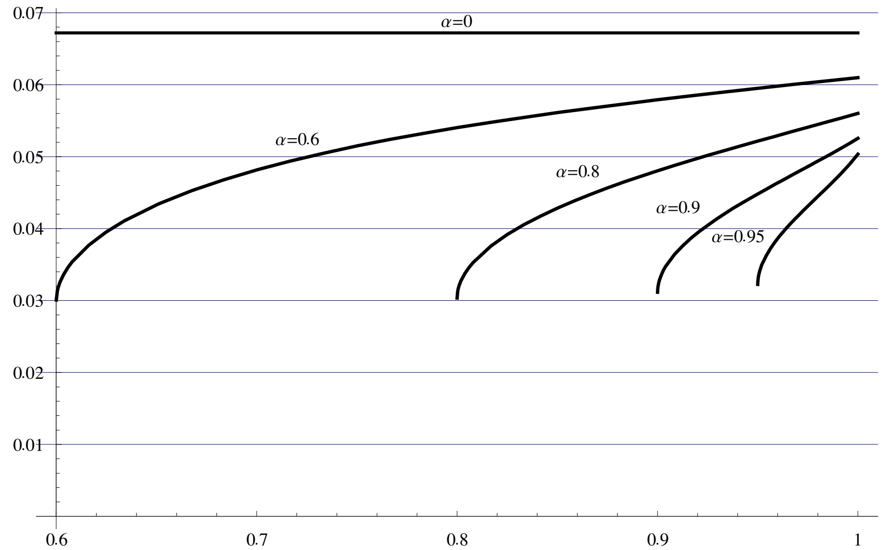

Figure 1.1 : Consumption-wealth ratio $\frac{c^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5$

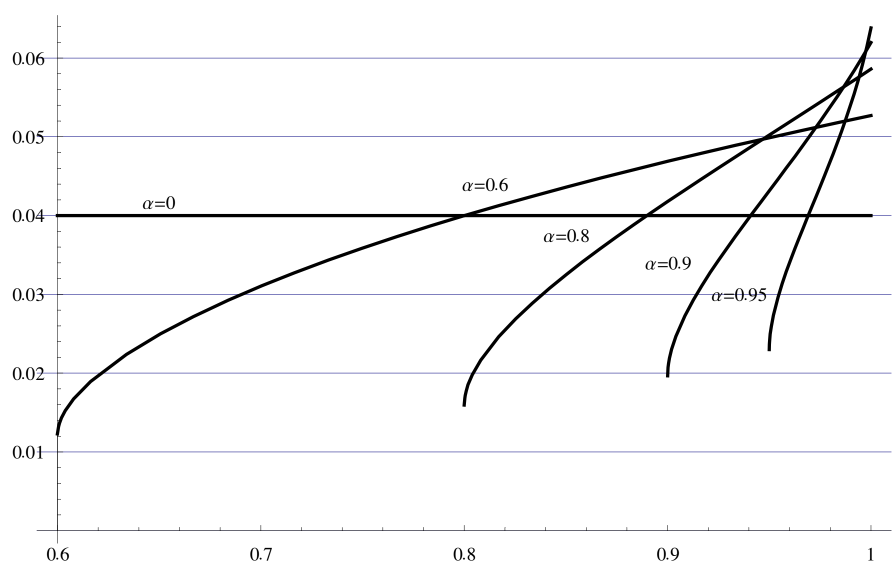

Figure 1.2 : Consumption-wealth ratio $\frac{c^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 0.8$

The consumption-wealth ratio $\frac{c^*}{W}$ is displayed in Figures 1.1 and 1.2 for several values of the drawdown constraint parameter α. For $b > 1$, as α goes up, $\frac{c^*}{W}$ uniformly shrinks and remains below the unconstrained ratio $\frac{1}{A} = 0.0672$. The reduction in consumption is large when wealth is close the minimum floor. For $\alpha = 0.6$, 0.8, 0.9 and 0.95, the endogenous ratchet coefficient for consumption α_c is 0.29, 0.43, 0.53 and 0.61 respectively. For $b < 1$, curves cross with one another and as asserted in proposition 2 when u is high enough, an increase in α leads to a higher consumption-wealth ratio that significantly exceeds the unconstrained ratio $\frac{1}{A} = 0.04$. When $\alpha = 0.6$, 0.8, 0.9 and 0.95, the values obtained for α_c are 0.14, 0.22, 0.28 and 0.34 respectively. Observe that larger drawdowns $1-\alpha_c$ from all-time high consumption level are allowed than in the case $b > 1$, reflecting the fact that the individual's IES is higher so she tolerates larger changes in her consumption plans across time.

We now examine the properties of the optimal portfolio strategy.

2.4.2 Assets allocations

The fraction of wealth invested in the risky asset is given by

$$\frac{z^*}{W} = \frac{\mu - r}{b\sigma^2}\left(1 - \frac{\beta_1 K_1(f'(u))^{\frac{\beta_1}{b}} + \beta_2 K_2(f'(u))^{\frac{\beta_2}{b}}}{A + K_1(f'(u))^{\frac{\beta_1}{b}} + K_2(f'(u))^{\frac{\beta_2}{b}}}\right).$$

The fraction of wealth invested in the risky asset is lower than in the unconstrained case, i.e. $\frac{\mu-r}{b\sigma^2}$. This is due in part to the hedging motives as described in the section 2.2. However, numerical simulations (displayed in the sequel) indicate that the investor's desire to dampen the ratchet effect plays a significant role in explaining the reduction in risky investment.

Proposition 3 *When $F_2(M,M) = 0$ is optimal, if $b > 1 (b < 1)$ and $\theta < \frac{1}{Y} (\theta > \frac{1}{Y})$, the fraction of wealth invested in the risky asset is non-decreasing in the ratio $\frac{W}{M}$; otherwise it is hump-shaped. When $z_1^* = 0$ is optimal, the fraction of wealth invested in the stock and the ratio $\frac{W}{M}$ are linked by an inverted U-relationship.*

Proof. See the Appendix. ■

Conditions for the logarithmic investor are more cumbersome and are presented in the Appendix.

Proposition 3 deserves several observations. First, choosing an increasing risky investment policy is optimal when the cost associated with the ratchet effect is not too large. Observe that θ is the consumption-wealth ratio at $u = 1$ for the myopic investor ($b = 1$). When $b > 1$, the investor´s IES is low ($s < 1$) and she is mainly concerned with the current consumption-wealth ratio and is reluctant to defer consumption. Proposition 3 suggests that the agent optimally chooses an increasing risky investment policy provided that at $u = 1$, $\frac{c^*}{M} = \frac{1}{Y}$ is above the corresponding value for the myopic investor. Conversely, when $b < 1$, when eager to defer consumption and to accept a low level of her current consumption-wealth ratio (below that of the myopic investor) at $u = 1$, the fraction of wealth invested in stocks is increasing. Nevertheless, note that since for $b > 1 (b < 1)$, at $u = 1$, the consumption-wealth ratio $\frac{c^*}{M} = \frac{1}{Y}$ goes down (up) when α increases forcing the investor to curb risky investment as a percentage of wealth.

Second, decreasing stock holdings as a percentage of wealth when W_t is close to M_t depart from the results obtained in Grossman and Zhou (1993) where the fraction of wealth invested in stock always increases in the ratio $\frac{W}{M}$. Recall that in Grossman and Zhou (1993) there is no intermediate consumption so intertemporal consumption substitution plays no role. Nevertheless, the hump-shaped relationship corroborates the intuition pointed out by these authors, i.e. αM is expected to grow at a faster rate than W and therefore investment in the risky asset is expected to fall. The lifetime utility relative risk aversion is no longer decreasing as (current) wealth rises but instead is U-shaped.

The condition for the ratio $\frac{z}{W}$ to be non-decreasing in $\frac{W}{M}$ depends on all the parameters of the model. A sufficient condition is $\theta < r (\theta > r)$ whenever $b > 1 (b < 1)$, i.e. the investor must be patient (impatient) enough when her relative risk aversion is high (low).

Proposition 4 *The more stringent the drawdown constraint (higher α), the smaller the fraction of wealth invested in the risky asset.*

Proof. See the Appendix. ■

Proposition 4 formally states that an increase in α *uniformly* reduces $\frac{z^*}{W}$ for all couples (W, M) and suggests that indeed risky investment is the favored channel to achieve wealth growth regulation.

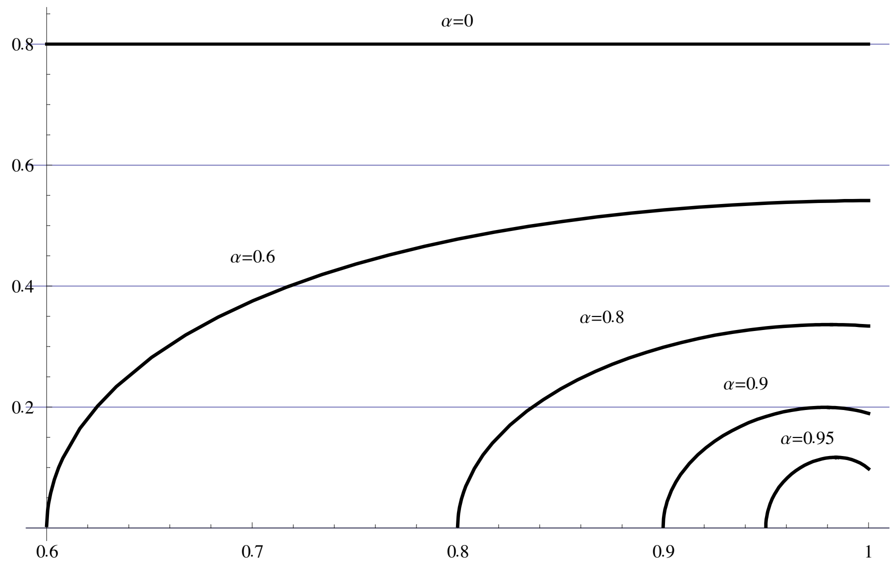

Figure 2 : Fraction of wealth invested in stocks $\frac{z^*}{W}$ as a function of u

$$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5$$

Figure 2 depicts the fraction of wealth invested in the risky asset $\frac{z^*}{W}$ for several values of the drawdown constraint parameter α. As α goes up, risky investment is reduced and when α is large enough, the curve $\frac{z^*}{W}$ is hump shaped. As a benchmark, when $\alpha = 0$, the unconstrained allocation the fraction $\frac{\mu - r}{b\sigma^2} = 0.8$. Indeed, observe that even when the current wealth W_t is far from the minimum floor αM_t, the reduction in stock holdings can be substantial.

Obviously, the analysis performed combined both hedging and ratchet effects. In order to disentangle the two effects, consider a fixed minimum floor equal to αM and compute the fraction of wealth invested in the stock $\frac{\widehat{z}^*}{W}$ when wealth W varies from αM up to M. Note that the ratio $\frac{\widehat{z}^*}{W}$ is independent of the choice of M.

Table I: Disentangling hedging and ratchet effects

α	u	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.925	0.95	0.975	1
0.6	$\frac{\widehat{z}^*}{W}$	0	0.283	0.383	0.450	0.499	0.537	0.568	0.581	0.592	0.603	0.613
	$\frac{z^*}{W}$	0	0.279	0.375	0.436	0.478	0.506	0.526	0.532	0.537	0.540	0.541
0.8	$\frac{\widehat{z}^*}{W}$	-	-	-	-	0	0.248	0.339	0.372	0.402	0.428	0.450
	$\frac{z^*}{W}$	-	-	-	-	0	0.230	0.299	0.318	0.331	0.336	0.334
0.9	$\frac{\widehat{z}^*}{W}$	-	-	-	-	-	-	0	0.168	0.234	0.283	0.322
	$\frac{z^*}{W}$	-	-	-	-	-	-	0	0.143	0.184	0.199	0.190

Table I reports stock holdings $\frac{\widehat{z}^*}{W}$ and those corresponding to the drawdown problem $\frac{z^*}{W}$ for several values of α. Recall that for the Merton Problem, this ratio is constant and equal to 0.8. Observe that hedging motives explain a significant share of the reduction in risky investment. Nevertheless, the ratchet effect becomes significant when the ratio $\frac{W}{M}$ approaches 1 and is enhanced as the drawdown constraint becomes more stringent (higher α). Taking the unconstrained portfolio allocation as a benchmark, at $u = 1$, the ratchet effect accounts for 9%, 14.5% and 16% for $\alpha = 0.6$, 0.8 and 0.9 respectively of the total reduction in stock holdings.

2.4.3 Representation of the optimal wealth process

Optimal policies (c^*, z^*) has been expressed in terms of state variables (W, M) using dynamic programming. Alternatively, it is possible to provide a representation in terms of simple regulated stochastic processes and gain some insights about the dynamics across time. Details of the derivation are presented in the Appendix.

First of all, we establish that the process $\frac{c^*}{M}$ is a two sided regulated geometric Brownian motion[9] with lower barrier $\frac{1}{X}$ and upper barrier $\frac{1}{Y}$ and for u in $(\alpha, 1)$, the dynamics are given by

$$d\left(\frac{c_t^*}{M_t}\right) = \frac{c_t^*}{M_t}\left((r - \frac{1}{A} + \frac{(\mu - r)^2}{b\sigma^2})dt + \frac{\mu - r}{b\sigma} dw_t\right).$$

[9] For the definition of a regulated Brownian motion, see Harisson (1985), p14.

Observe that this law of motion is the same as the one that governs the optimal consumption process in the Merton problem. Then, the representation of current wealth W, all time maximum wealth M and consumption c^* as stochastic processes depends on the boundary condition at $u = 1$. When $F_2(M, M) = 0$ is optimal, we show that the process $\log H\left(\frac{W_t}{M_t}\right)$ is a one sided regulated arithmetic Brownian motion with lower barrier $-\log Y$, and for u in $(\alpha, 1)$,

$$d\log H\left(\frac{W_t}{M_t}\right) = (\frac{r-\theta}{b} + \frac{(\mu-r)^2}{2b\sigma^2})dt + \frac{\mu-r}{b\sigma}dw_t.$$

If $z_1^* = 0$ is optimal, when wealth hits its peak for the first time τ_0, we have

$$dW_t = (r - \frac{1}{Y})W_t dt.$$

There are two cases. If $\frac{1}{Y} < r$, wealth will keep on increasing forever and for all $t \geq \tau_0$, $W_t \equiv M_t$, $c_t^* \equiv \frac{M_t}{Y}$ and $z_t \equiv 0$. The ceiling $W = M$ is an upper absorbing barrier. Recall that $Y \leq \frac{b}{\theta+(b-1)r}$ $\left(Y \geq \frac{b}{\theta+(b-1)r}\right)$ whenever $b \leq 1$ $(b \geq 1)$. Thus, a sufficient condition to have an upper absorbing barrier when $b \geq 1$ is $\theta \leq r$, i.e. when the IES is small, the time discount rate needs to be smaller than the riskfree rate. Conversely, if $r \leq \frac{1}{Y}$, wealth is driven down immediately after hitting its peak and cannot exceed M_0. The ceiling $W = M$ is an upper reflecting barrier. A sufficient condition to have an upper reflecting barrier when $b \leq 1$ is $\theta \geq r$, i.e. when the IES is large, the time discount rate needs to be larger than the riskfree rate.

In the next section, we estimate the cost induced by the constraint.

3 COST OF THE DRAWDOWN CONSTRAINT

There are several ways of estimating the cost of the drawdown constraint. We can assess the loss in terms of forgone lifetime utility; alternatively, we can measure it in terms of the numéraire. We start with the first measure and to keep things simple, we derive the maximum cost when $\alpha = 1$.

3.1 Cost in terms of forgone lifetime utility

Wealth must always be maintained at its maximum, so in order not to violate the drawdown constraint, holdings in the stock must be zero $z^* \equiv 0$. This is equivalent to solving the deterministic optimal consumption-portfolio problem when only a bond is available. The optimal level of consumption is proportional to wealth with $c = \frac{W}{A_0}$ and the evolution of wealth is deterministic

$$dW_t = \frac{r-\theta}{b}W_t dt.$$

The problem is well defined if and only if $r \geq \theta$ and the corresponding value function is $\frac{A_0^b W^{1-b}}{1-b}$. The (maximum) cost of the drawdown constraint in terms of loss of the lifetime utility is the relative difference between the constrained and unconstrained value functions. For $b \neq 1$, it is simply given by

$$\left(\frac{A_0}{A}\right)^{\varepsilon b} - 1,$$

where $\varepsilon = 1(\varepsilon = -1)$ if $b > 1(b < 1)$. For $\mu = 0.12$, $r = 0.06$, $\sigma = 0.2$, $\theta = 0.05$, $b = 2.5$, the loss is approximately 55.4%. It decreases with the instantaneous variance σ^2 but increases with the mean return μ.

3.2 Cost in terms of the numéraire

We calculate the percentage k increase in wealth necessary to bring the level of the lifetime utility to the level of those of an unconstrained investor, i.e. we want to determine k such that $F((1+k)W) = \frac{A^b}{1-b}W^{1-b}$. We obtain

$$k = \left(\frac{A_0}{A}\right)^{\frac{b}{1-b}} - 1,$$

and for the parameters chosen previously, we find that the percentage increase k is approximately 34.2%. It also decreases with the instantaneous variance σ^2 and increases with the mean return μ.

For both measures, the cost induced by the constraint is economically significant.

4 EXTENSION OF THE BASIC MODEL

In this section, we consider the case of an agent who derives utility from current consumption and also from her status. Broadly speaking, there are two rival theories of social status: ascription versus achievement. Individual position can be ascribed by virtue of their age, sex, race, and family membership or connection. Alternatively, individuals can achieve their own position by their own performance and merits. Here, we interpret a society in which higher wealth confers a higher status. Status can confer power, privileges, access to political circles or social events, and at a more personal level, enhance self esteem. As argued in Cole, Mailath, and Postlewaite (1992), social status can determine the degree of success one group member may have with non-market decisions such as finding a good mate for instance. Weber (1968) refers to a status group as a collection of individuals who happen to have a common lifestyle and share the same economic interest. Maintaining one's membership of a status group is certainly desirable and ambition may dictate social climbing; however, individuals may be reluctant to lower their position in society. People who experience a downward social shift may experience depression or poor psychological well being.[10]

Following Bakshi and Chen (1996) and Smith (2001), to keep things simple, we retain current wealth level W as an index of status and status seeking is modeled as direct preference for financial wealth. More specifically, preferences[11] are given by

$$u(c, W, M) = \begin{cases} \frac{(c+aW)^{1-b}}{1-b}, & W \geq \alpha M \\ -\infty, & \text{otherwise}, \end{cases}$$

where parameter $a > 0$ governs how much the agent cares about her social status.

We first examine the optimal consumption-portfolio choices for an unconstrained investor.

4.1 Benchmark Case

In the absence of status downfall fear, the agent aims at maximizing her lifetime utility

$$F(W_t) = \max_{(c,z)} E_t \left[\int_t^{\infty} \frac{(c_s + aW_s)^{1-b}}{1-b} e^{-\theta(s-t)} ds \right],$$

subject to the budget constraint (1) with $W_t > 0$ given. The transversality condition for this problem is the same as in the Merton problem. The Hamilton Jacobi Bellman (HJB) equation of this problem is

$$\theta F = \max_{(c,z)} \frac{(c + aW)^{1-b}}{1-b} + (rW - c + z(\mu - r)) F' + \frac{\sigma^2}{2} z^2 F''.$$

The optimal conditions are

$$c^* = (F')^{-\frac{1}{b}} - aW \text{ and } z^* = -\frac{(\mu - r)F'}{\sigma^2 F''},$$

and F satisfies the following non-linear ODE

$$\theta F = \frac{b(F')^{\frac{b-1}{b}}}{1-b} + (r + a)WF' - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \frac{(F')^2}{F''}. \tag{11}$$

[10]The University of Newcastle upon Tyne study by Parker, Pearce and Tiffin (2005) indicates that women are twice as likely to be downwardly mobile. The study involved men and women born in 1947 in Newcastle and followed them from childhood to age 50. Researchers noted the findings might be explained by the fact that men born during that era gained much of their self-esteem from their careers, whereas women found fulfillment from social pursuits outside of work, such as children and friendships. It's also possible that women are more emotionally resilient in such situations, the researchers suggested.

[11]The choice of the functional form of the utility function is motivated by tractability reasons.

Inspection of the ODE (11) reveals that the problem can be nested in a standard Merton problem as in section 2.1 with a riskfree rate $r + a$ and a mean return of the stock $\mu + a$. It follows that the optimal allocations are given by

$$\begin{aligned} \frac{c^*}{W} &= \frac{1}{\widetilde{A}} - a \\ \frac{z^*}{W} &= \frac{\mu - r}{b\sigma^2}, \end{aligned}$$

with

$$(\widetilde{A})^{-1} = \frac{\theta}{b} + \frac{b-1}{b}\left(r + a + \frac{(\mu - r)^2}{2b\sigma^2}\right) > 0.$$

For our choice of the functional form of the utility function, the lifetime utility relative risk aversion is constant and equal to b so the risky portfolio strategy is unchanged by the spirit of capitalism. Status seeking only affects the optimal consumption plans. Recall that the agent derives utility through two channels: current consumption c and current wealth W. These two channels compete with each other: the higher the consumption, the lower wealth accumulation and therefore the lower the future status. An increase in status enjoyment (higher a) leads to a decrease in the consumption-wealth ratio: the agent chooses to foster wealth accumulation, which is reflected by a higher mean growth of the wealth process.

We now study the case when the agent is reluctant to accept large status downfalls. In particular, we will see that status has a significant impact on stock holdings.

4.2 Maintaining Social Status

Given the homogeneity of degree $1-b$ in (c, W) of the utility function, the linearity in variables (W, M) of the drawdown constraint (2) and the form of the "reduced" HJB (11), it is easy to realize that the analysis performed for the case $a = 0$ still applies if we substitute $\frac{1}{A}$ with $\frac{1}{\widetilde{A}}$ and replace (μ, r) with $(\mu + a, r + a)$ in the definition of roots β_1 and β_2.

We now investigate the quantitative impact of status on the optimal allocations.

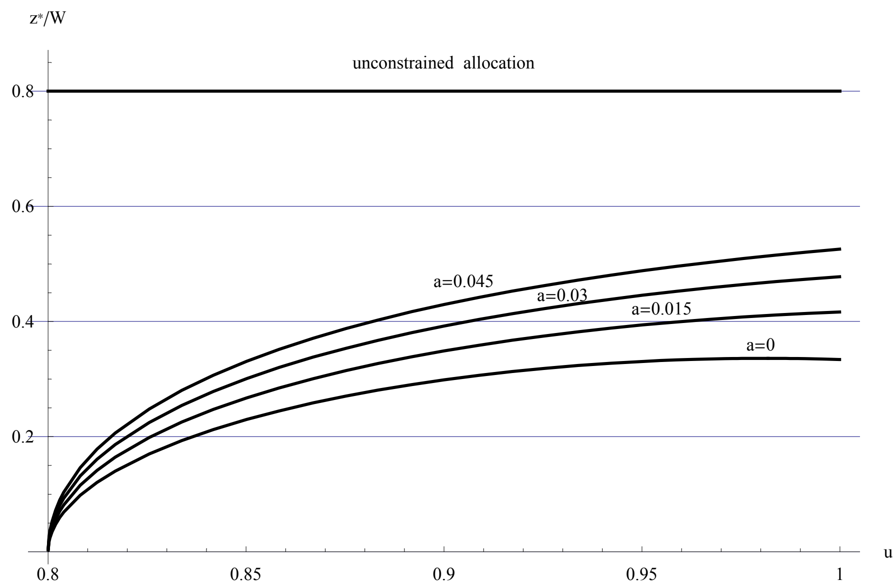

Figure 3 : Fraction of wealth invested in stocks $\frac{z^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5,\ \alpha = 0.8$

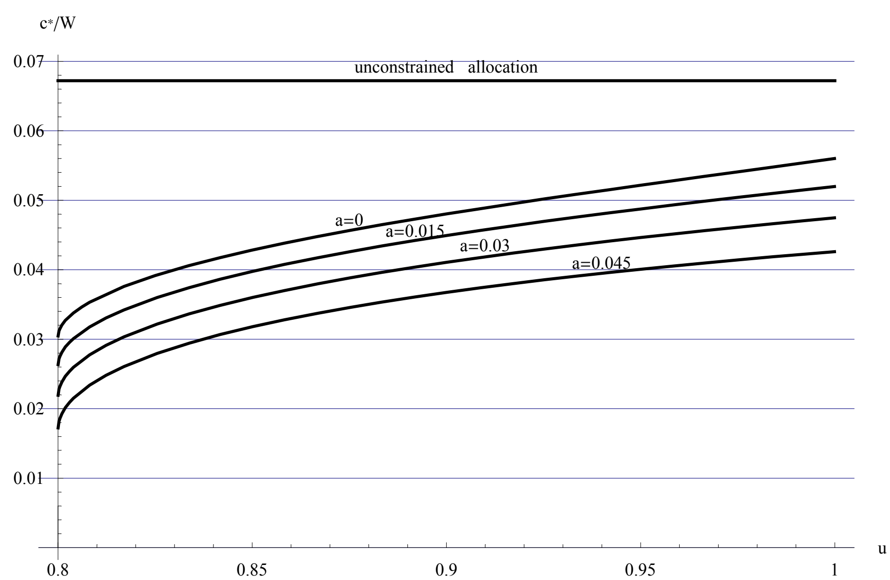

Figure 4 : Consumption-wealth ratio $\frac{c^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5,\ \alpha = 0.8$

Figures 3 and 4 represent the fraction of wealth invested in the risky asset $\frac{z^*}{W}$ and the consumption-wealth ratio $\frac{c^*}{W}$ for several values[12] of the parameter a. As a goes up, the investor increasingly values social status, which leads to heavier stock holdings in an attempt to achieve a higher growth rate of her wealth (Figure 3). In parallel, observe in Figure 4 that the consumption-wealth ratio uniformly shrinks. Priority shifts towards building up status that has a persistent impact whereas consumption is less appealing since its effect is only instantaneous. When a ratcheting behavior is imposed on wealth, introducing wealth as a proxy for social status in the utility function fosters risky investment in a substantial manner.

5 CONCLUSION

We have examined the implications of the intolerance of a large decline in wealth on optimal consumption and portfolio policies for an investor with constant relative risk aversion preferences. We find that wealth ratcheting induces a ratcheting behavior of consumption as current optimal consumption is always maintained above a fixed percentage of its all-time maximum. Hedging motives and mitigating the cost associated with the ratchet feature of the constraint govern the agent's intertemporal choices. We have isolated the impact of hedging by analyzing asset management for a foundation required to preserve its endowment. Essentially, the lifetime utility relative risk aversion rises, which leads to smaller stock holdings. Looking at the wedge between optimal allocations for the fixed floor and a ratchet floor allows us to quantify the ratchet impact and uncover its significance. In particular, the investor may curb investment in stocks as wealth approaches its peak to limit its growth and the risk of raising the minimum floor. An extension of the basic model incorporates the spirit of capitalism and interprets wealth as an index for social status. Lasting benefits from current and future status levels provide incentives for a higher growth of the wealth and induce a more aggressive risky investment strategy at the expense of consumption.

Another possible extension would be to include labor income. If the correlation between labor income and the stock market is small or negative, the investor naturally would like to borrow against her future income to increase risky investment, which could drive down her financial wealth and exacerbate both hedging motives and the ratchet effect. A detailed analysis is left for future research.

[12] For higher values of a, the reflecting condition at the minimum floor αM is violated.

6 APPENDIX

A.1. Proof of Lemma 1

Consider the duality approach and the following changes of variables: $X = F'(W), W = -J'(X)$ and $F(W) = J(X) - XJ'(X)$. Using relationship (7), we find that the function J must be solution of the following linear ODE

$$\theta' J(X) = \frac{bX^{\frac{b-1}{b}}}{1-b} + (\theta' - r')XJ'(X) + \frac{1}{2}\left(\frac{\mu' - r'}{\sigma}\right)^2 X^2 J''(X).$$

The general solution is

$$J(X) = \frac{bAX^{\frac{b-1}{b}}}{1-b} + \frac{bL_1}{\beta_1' - 1 - b} X^{\frac{\beta_1' - 1 - b}{b}} + \frac{bL_2}{\beta_2' - 1 - b} X^{\frac{\beta_2' - 1 - b}{b}}, \tag{12}$$

where L_1 and L_2 are constants. Differentiating (12) with respect to X and using the fact that $X = F'(W)$ and $W = -J'(X)$ provides the desired result. ■

A.2. Proof of Properties $\beta_1' > 1$ and $1 - b - \beta_2' > 0$.

Recall that β_1' is the positive root of the quadratic

$$Q(x) = \frac{1}{2}\left(\frac{\mu' - r'}{b\sigma}\right)^2 x^2 + \left(\frac{1}{A} - r' - \frac{1}{2}\left(\frac{\mu' - r'}{b\sigma}\right)^2\right) x - \frac{1}{A}.$$

Since $Q(1) = -r' < 0$, we must have $\beta_1' > 1$. Then, using the fact that

$$\frac{1}{2}\left(\frac{\mu' - r'}{b\sigma}\right)^2 A\beta_2'\beta_1' = 1 \text{ and } \frac{1}{2}\left(\frac{\mu' - r'}{b\sigma}\right)^2 (\beta_1' + \beta_2') = -\frac{1}{A} + r' + \frac{1}{2}\left(\frac{\mu' - r'}{b\sigma}\right)^2,$$

we find that

$$(\beta_1' + b - 1)(1 - b - \beta_2') = \frac{\theta'}{\frac{1}{2}\left(\frac{\mu' - r'}{b\sigma}\right)^2}.$$

Since $\beta_1' > 1$, indeed we have $1 - b - \beta_2' > 0$. ■

A.3. Proof of Properties P1 and P2

P1. F is strictly increasing in W and decreasing in M since given W, the higher M, the more stringent the drawdown constraint. Let $\lambda \in (0, 1)$, (W_0, M_0) and (W_0', M_0) be two initial states and $(c, (x, z))$ and $(c', (x', z'))$ the associated optimal strategies. Then, for initial wealth $\lambda W_0 + (1 - \lambda)W_0'$, $(\lambda c + (1 - \lambda)c', \lambda x + (1 - \lambda)x', \lambda z + (1 - \lambda)z')$ is also a feasible strategy as the wealth dynamics are linear in variables (c, x, z) and

$$\begin{aligned} \lambda W_t + (1 - \lambda)W_t' &\geq \lambda \alpha M_t + (1 - \lambda)\alpha M_t \\ &\geq \alpha \max\{M_0, \lambda W_s + (1 - \lambda)W_s', s \leq t\}. \end{aligned}$$

Finally, by strict concavity of the utility function u

$$E_0\left[\int_0^\infty u(\lambda c_s + (1-\lambda)c'_s)e^{-\theta s}ds\right] > E_0\left[\int_0^\infty \Big(\lambda u(c_s) + (1-\lambda)u(c'_s)\Big)\, e^{-\theta s}ds,\right.$$

which implies that $F(\lambda W_0 + (1-\lambda)W'_0, M_0) > \lambda F(W_0, M_0) + (1-\lambda)F(W'_0, M_0)$. ■

P2. Let $(c,(x,z))$ be feasible for an initial state (W_0, M_0) and $\lambda \in (0,1)$. Then $(\lambda c, (\lambda x, \lambda z))$ is feasible for the initial state $(\lambda W_0, \lambda M_0)$ since the dynamics of the corresponding wealth process W_λ are

$$\begin{aligned} dW_\lambda(s) &= (\lambda r x_s - \lambda c_s + \lambda z_s \mu)ds + \lambda z_s \sigma dw_s \\ &= \lambda dW_s, \end{aligned}$$

so $W_\lambda(s) = \lambda W_s$ and therefore $W_\lambda(s) = \lambda W_s \geq \alpha\lambda M_s = \alpha M_\lambda(s)$. It follows that $F(\lambda W, \lambda M) \leq \lambda^{1-b}F(W,M)$ by homogeneity of degree $1-b$ the utility function. Finally

$$F(W,M) = F(\lambda^{-1}\lambda W, \lambda^{-1}\lambda M) \leq \lambda^{b-1}F(\lambda W, \lambda M),$$

so in fact we have $F(\lambda W, \lambda M) = \lambda^{1-b}F(W,M)$. ■

A.4. Derivation of Boundary Conditions

Condition for a well defined value function. The boundary conditions must be such that f is well defined and the drawdown constraint is met. Taking derivatives with respect to u relationship (8), it is easy to see that for all u in $(\alpha, 1)$

$$-\frac{bf'(u)}{f''(u)} = A(f'(u))^{-\frac{1}{b}} - (\beta_1 - 1)K_1(f'(u))^{\frac{\beta_1-1}{b}} - (\beta_2 - 1)K_2(f'(u))^{\frac{\beta_2-1}{b}}. \tag{13}$$

Since $-\frac{bf'(u)}{f''(u)}$ is non-positive, for all u in $[\alpha, 1]$, we must have

$$(\beta_1 - 1)K_1(f'(u))^{\frac{\beta_1}{b}} + (\beta_2 - 1)K_2(f'(u))^{\frac{\beta_2}{b}} \leq A. \tag{14}$$

Then, set $Y = (f'(1))^{\frac{1}{b}}$ and $X = (f'(\alpha))^{\frac{1}{b}}$ and notice that $Y \leq X$. Given relationship (8), it must be the case that the function

$$\begin{aligned} & [Y,X] && \rightarrow && \mathbb{R} \\ \Phi: \;& y && \mapsto && Ay^{-1} + K_1 y^{\beta_1 - 1} + K_2 y^{\beta_2 - 1}, \end{aligned}$$

is invertible so we can write $f'(u) = \left(\Phi^{-1}(u)\right)^b$, for all u in $[\alpha, 1]$. Since f'' is negative, then Φ' must be negative. Condition (14) is equivalent to

$$\Psi(y) = -A + (\beta_1 - 1)K_1 y^{\beta_1} + (\beta_2 - 1)K_2 y^{\beta_2} < 0.$$

As shown in the sequel, we must have $\Psi(X)=0$. For $K_1>0$ and $K_2<0$ (to be justified later), it turns out that

$$\Psi'(y)=\beta_1(\beta_1-1)K_1y^{\beta_1-1}+\beta_2(\beta_2-1)K_2y^{\beta_2-1},$$

is strictly increasing and has at most one root on $[Y,X]$. Hence, the condition $\Psi(Y)\leq 0$ is necessary and sufficient to guarantee that Ψ is negative on (X,Y). ■

Boundary condition at $u=1$. First of all, we have

$$1=A(f'(1))^{-\frac{1}{b}}+K_1(f'(1))^{\frac{\beta_1-1}{b}}+K_2(f'(1))^{\frac{\beta_2-1}{b}}. \tag{15}$$

Then, for $h>0$, over the interval of time $[t,t+h]$, the HJB is

$$F(W_t,M_t)=\max_{(c,z)}E_t\left[\frac{c_t^{1-b}}{1-b}+e^{-\theta h}F(W_{t+h},M_{t+h})\right],$$

so using Ito lemma for semi-martingales

$$\begin{aligned}0= &\max_{(c,z)} \frac{c_t^{1-b}}{1-b}+E_t\left[\int_t^{t+h}\left(-\theta F+(rW-c+z(\mu-r))F_1+\tfrac{\sigma^2}{2}z^2F_{11}\right)ds\right]\\ &+E_t\left[\int_t^{t+h}F_2dM\right].\end{aligned}$$

As derived in Grossman and Zhou (1993)

$$E_t\left[M_{t+h}-M_t\mid W_t=M_t\right]=\sqrt{\frac{2}{\pi}}\sigma\left|z\right|\sqrt{h}+O(h).$$

When h is small, $\sqrt{h}$ dominates h so in order for the Bellman equation to hold at $W=M$, we must have

$$F_2(M,M)=0 \text{ or } z_1^*=0,$$

$\mathrm{F}_2(M,M)=0$ or $z_1^*=0$,and the HJB (7) is also valid for $u=1$. Since the definition of the HJB involves a maximization over z, *whenever feasible*, it is optimal to choose $z_1^*=-\frac{(\mu-r)f'(1)}{\sigma^2f''(1)}$, instead of $z_1^*=0$.

Case 1: $z_1^*>0$ **is optimal.** We must have $F_2(M,M)=0$ or equivalently $f'(1)=(1-b)f(1)$. Using relationship (7) leads to

$$\begin{aligned}\frac{\theta}{1-b} = &\left(\frac{b}{1-b}+(r+\frac{1}{2b}\left(\frac{\mu-r}{\sigma}\right)^2)A\right)(f'(1))^{-\frac{1}{b}}+\left(r-\frac{\beta_1-1}{2b}\left(\frac{\mu-r}{\sigma}\right)^2\right)K_1(f'(1))^{\frac{\beta_1-1}{b}}\\ &+\left(r-\frac{\beta_2-1}{2b}\left(\frac{\mu-r}{\sigma}\right)^2\right)K_2(f'(1))^{\frac{\beta_2-1}{b}}.\end{aligned}$$

Using the definition of A, relationship (15) and the fact that β_1 and β_2 are the roots of the quadratic (9), it follows that

$$-\frac{A(f'(1))^{-\frac{1}{b}}-1}{A}=\frac{b-1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2\left(\beta_1K_1(f'(1))^{\frac{\beta_1-1}{b}}+\beta_2K_2(f'(1))^{\frac{\beta_2-1}{b}}\right),$$

and finally since $\beta_1\beta_2=-\frac{1}{\frac{A}{2}\left(\frac{\mu-r}{b\sigma}\right)^2}$, we find that

$$-\beta_1\beta_2\left(A(f'(1))^{-\frac{1}{b}}-1\right)=(1-b)\left(\beta_1K_1(f'(1))^{\frac{\beta_1-1}{b}}+\beta_2K_2(f'(1))^{\frac{\beta_2-1}{b}}\right).\ \blacksquare$$

Case 2: $z_1^*=0$ is optimal. From the HJB equation, we have

$$\theta f(1)=\frac{b(f'(1))^{\frac{b-1}{b}}}{1-b}+rf'(1).\ \blacksquare$$

Boundary condition at $u=\alpha$. At $W=\alpha M$, risky investment must be zero in order not to violate the constraint in the near future with some positive probability. From relationship (13), we find that

$$\frac{z^*}{W}=\frac{\mu-r}{b\sigma^2}\left(1-\frac{\beta_1K_1(f'(u))^{\frac{\beta_1-1}{b}}+\beta_2K_2(f'(u))^{\frac{\beta_2-1}{b}}}{A(f'(u))^{-\frac{1}{b}}+K_1(f'(u))^{\frac{\beta_1-1}{b}}+K_2(f'(u))^{\frac{\beta_2-1}{b}}}\right).$$

At $u=\alpha$, we have

$$\alpha=A(f'(\alpha))^{-\frac{1}{b}}+K_1(f'(\alpha))^{\frac{\beta_1-1}{b}}+K_2(f'(\alpha))^{\frac{\beta_2-1}{b}},$$

and $z_\alpha^*=0$ implies

$$A=(\beta_1-1)K_1(f'(\alpha))^{\frac{\beta_1}{b}}+(\beta_2-1)K_2(f'(\alpha))^{\frac{\beta_2}{b}}.$$

To summarize, the boundary conditions are:

$$\begin{aligned}
&\alpha X=A+K_1X^{\beta_1}+K_2X^{\beta_2}\\
&A=(\beta_1-1)K_1X^{\beta_1}+(\beta_2-1)K_2X^{\beta_2}\\
&Y=A+K_1Y^{\beta_1}+K_2Y^{\beta_2}\\
&\begin{cases}\beta_1\beta_2\left(Y-A\right)=(1-b)\left(\beta_1K_1Y^{\beta_1}+\beta_2K_2Y^{\beta_2}\right) & \text{if } z_1^*>0\\ A=(\beta_1-1)K_1Y^{\beta_1}+(\beta_2-1)K_2Y^{\beta_2} & \text{if } z_1^*=0.\end{cases}
\end{aligned}$$

Using the fact that $A_0=-\frac{\beta_1\beta_2A}{1-b-\beta_1\beta_2}$, the system can be rewritten as stipulated in the core of the paper. $\blacksquare$

A.5. Proof of Proposition 1

We examine the condition $\Psi(Y)\leq 0$ derived in A.4. When $F_2(M,M)=0$ we must have

$$(\beta_1-1)K_1Y^{\beta_1}+(\beta_2-1)K_2Y^{\beta_2}\leq A.$$

Since in this case

$$A - (\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2} = \frac{1 - b - \beta_1\beta_2}{b - 1}(A_0 - Y),$$

it follows that $Y \leq A_0$ ($Y \geq A_0$) if $b \geq 1 (b \leq 1)$. When $z_1^* = 0$ is optimal, note that $A_0 \geq A (A_0 \leq A)$ whenever $b \geq 1 (b \leq 1)$, so in this case, we have $Y \geq A (Y \leq A)$ whenever $b \geq 1 (b \leq 1)$. ■

A.6.1. Proof of Existence and Uniqueness of (X, Y, K_1, K_2) when $z_1^* > 0$

We want to show existence and uniqueness for the following 4 by 4 non-linear system:

$$\begin{aligned} \alpha X &= A + K_1 X^{\beta_1} + K_2 X^{\beta_2} && (16) \\ A &= (\beta_1 - 1)K_1 X^{\beta_1} + (\beta_2 - 1)K_2 X^{\beta_2} && (17) \\ Y &= A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2} && (18) \\ \beta_1\beta_2 (Y - A) &= (1 - b)\left(\beta_1 K_1 Y^{\beta_1} + \beta_2 K_2 Y^{\beta_2}\right). && (19) \end{aligned}$$

Combining relationships (18) and (19) leads to

$$\beta_1(1 - b - \beta_2)K_1 Y^{\beta_1} + \beta_2(1 - b - \beta_1)K_2 Y^{\beta_2} = 0.$$

Since both $\beta_1(1 - b - \beta_2)$ and $\beta_2(1 - b - \beta_1)$ are positive, it must be the case that K_1 and K_2 have opposite signs. Then

$$K_1 Y^{\beta_1} = \frac{\beta_2(1 - b - \beta_1)}{\beta_1 - \beta_2}\frac{Y - A}{b - 1},$$

which implies that K_1 has the same sign as $\frac{Y-A}{b-1}$. Eliminating K_1 and K_2 from relationship (17) yields

$$A = \left(\beta_2(\beta_1 - 1)(1 - b - \beta_1)\left(\frac{X}{Y}\right)^{\beta_1} - \beta_1(\beta_2 - 1)(1 - b - \beta_2)\left(\frac{X}{Y}\right)^{\beta_2}\right)\frac{Y - A}{(b - 1)(\beta_1 - \beta_2)}.$$

Since both $\beta_2(\beta_1 - 1)(1 - b - \beta_1)$ and $-\beta_1(\beta_2 - 1)(1 - b - \beta_2)$ are positive, we find that $\frac{Y-A}{b-1}$ is indeed positive. Hence $K_1 > 0$ **and** $K_2 < 0$ and $Y \geq A$ ($Y \leq A$) exactly when $b \geq 1$ ($b \leq 1$). Combining (16) and (17) leads

$$\alpha X = \beta_1 K_1 X^{\beta_1} + \beta_2 K_2 X^{\beta_2},$$

and eliminating K_1 and K_2 using relationships (18) and (19) yields

$$\alpha X = \beta_1\beta_2\left((1 - b - \beta_1)\left(\frac{X}{Y}\right)^{\beta_1} - (1 - b - \beta_2)\left(\frac{X}{Y}\right)^{\beta_2}\right)\frac{Y - A}{(b - 1)(\beta_1 - \beta_2)}.$$

Set $\varpi = \frac{X}{Y} \geq 1$, we have

$$X = \frac{\beta_1\beta_2 A\left((1 - b - \beta_1)\varpi^{\beta_1} - (1 - b - \beta_2)\varpi^{\beta_2}\right)}{\alpha\left(\beta_2(\beta_1 - 1)(1 - b - \beta_1)\varpi^{\beta_1} - \beta_1(\beta_2 - 1)(1 - b - \beta_2)\varpi^{\beta_2}\right)},$$

and ϖ is implicitly defined by

$$\begin{array}{rcl}\alpha(b-1)(\beta_1-\beta_2) & = & -\alpha\left(\beta_2(\beta_1-1)(1-b-\beta_1)\varpi^{\beta_1}-\beta_1(\beta_2-1)(1-b-\beta_2)\varpi^{\beta_2}\right) \\ & & +\beta_1\beta_2\left((1-b-\beta_1)\varpi^{\beta_1-1}-(1-b-\beta_2)\varpi^{\beta_2-1}\right).\end{array}$$

Define the auxiliary function

$$\begin{array}{rlcl} & [1,\infty) & \to & \mathbb{R} \\ \Phi: & x & \mapsto & \alpha\left(\beta_2(\beta_1-1)(1-b-\beta_1)x^{\beta_1}-\beta_1(\beta_2-1)(1-b-\beta_2)x^{\beta_2}\right) \\ & & & -\beta_1\beta_2\left((1-b-\beta_1)x^{\beta_1-1}-(1-b-\beta_2)x^{\beta_2-1}\right)-\alpha(1-b)(\beta_1-\beta_2).\end{array}$$

We want to show that Φ has a unique root $\varpi>1$. Φ is continuously differentiable and $\Phi(1)=(1-\alpha)\beta_1\beta_2(\beta_1-\beta_2)<0$. Then, we show that $\Phi(\frac{1}{\alpha})<0$. A little bit of algebra yields

$$\Phi(\frac{1}{\alpha})=-\beta_2(1-b-\beta_1)\left(\frac{1}{\alpha}\right)^{\beta_1-1}+\beta_1(1-b-\beta_2)\left(\frac{1}{\alpha}\right)^{\beta_2-1}-\alpha(1-b)(\beta_1-\beta_2).$$

Define an auxiliary function

$$\begin{array}{rlcl} & (1,\infty) & \to & \mathbb{R} \\ \Theta: & y & \mapsto & -\beta_2(1-b-\beta_1)y^{\beta_1}+\beta_1(1-b-\beta_2)y^{\beta_2}.\end{array}$$

Again Θ is continuous and differentiable and $\lim_{1}\Theta=(1-b)(\beta_1-\beta_2)$. Clearly, Θ is decreasing, which implies that for all y in $(1,\infty)$, $\Theta(y)<(1-b)(\beta_1-\beta_2)$ and in particular $\Phi(\frac{1}{\alpha})<0$. Then

$$\Phi'(x)=-\beta_1\beta_2x^{\beta_2-2}(\alpha x-1)\Psi(x),$$

where $\Psi(x)=-(\beta_1-1)(1-b-\beta_1)x^{\beta_1-\beta_2}+(\beta_2-1)(1-b-\beta_2)$. Since $-(\beta_1-1)(1-b-\beta_1)>0$ and $\beta_1-\beta_2>0$, g is strictly increasing and

$$\Psi(1)=-(\beta_1-1)(1-b-\beta_1)+(\beta_2-1)(1-b-\beta_2).$$

Case 1: $\Psi(1)>0$. In this case, Ψ is strictly positive and therefore Φ is decreasing on $\left[1,\frac{1}{\alpha}\right]$ and increasing on $[\frac{1}{\alpha},\infty)$. Since Φ is continuous and $\lim_{\infty}\Phi=\infty$ we conclude that Φ has a unique root $\varpi\in[\frac{1}{\alpha},\infty)$. ■

Case 2: $\Psi(1)<0$. Then define x^* such that $\Psi(x^*)=0$, i.e.

$$x^*=\left(\frac{(\beta_2-1)(1-b-\beta_2)}{(\beta_1-1)(1-b-\beta_1)}\right)^{\frac{1}{\beta_1-\beta_2}}.$$

It follows that Ψ is negative on $[1, x^*]$ and positive on $[x^*, \infty)$. Then we need to distinguish whether x^* is smaller or greater than $\frac{1}{\alpha}$.

Case 2.1: $\frac{1}{\alpha} < x^*$. In this case, Φ is increasing on $\left[1, \frac{1}{\alpha}\right]$, decreasing on $\left[\frac{1}{\alpha}, x^*\right]$ and finally increasing on $[x^*, \infty)$. Since $\Phi(\frac{1}{\alpha}) < 0$, we conclude that Φ has a unique root that belongs to $[x^*, \infty)$. $\blacksquare$

Case 2.2: $x^* < \frac{1}{\alpha}$. In this case, Φ is increasing on $[1, x^*]$, then decreasing on $\left[x^*, \frac{1}{\alpha}\right]$ and finally increasing on $[\frac{1}{\alpha}, \infty)$. It remains to show that $\Phi(x^*) < 0$ to conclude that Φ has a unique root that belongs to the interval $[\frac{1}{\alpha}, \infty)$. Using the definition of x^*, one can show that

$$\Phi(x^*) = (\beta_1 - \beta_2)(\beta_1 + b - 1)\left(\alpha(\beta_1 - 1)(x^*)^{\beta_1} - \frac{\beta_1\beta_2}{\beta_2 - 1}(x^*)^{\beta_1 - 1}\right) - \alpha(1 - b)(\beta_1 - \beta_2).$$

Define an auxiliary function

$$\Xi : \begin{array}{ccc} (1, \infty) & \rightarrow & \mathbb{R} \\ y & \mapsto & \alpha(\beta_1 - 1)y^{\beta_1} - \frac{\beta_1\beta_2}{\beta_2 - 1}y^{\beta_1 - 1}, \end{array}$$

Again Ξ is continuous and differentiable and

$$\Xi'(y) = \beta_1(\beta_1 - 1)y^{\beta_1 - 2}(\alpha y - \frac{\beta_2}{\beta_2 - 1}).$$

It follows that Ξ' is negative on $\left[1, \frac{\beta_2}{\alpha(\beta_2 - 1)}\right]$ and then increasing on $\left[\frac{\beta_2}{\alpha(\beta_2 - 1)}, \frac{1}{\alpha}\right]$. Consequently, since $(\beta_1 - \beta_2)(\beta_1 + b - 1) > 0$, it must be the case that $\Phi(x^*) \leq \max\{\Phi(1), \Phi(\frac{1}{\alpha})\}$, so in particular $\Phi(x^*) < 0$ and the desired result follows.

To summarize, there is a unique real number $\varpi > \frac{1}{\alpha}$ such that $\Phi(\varpi) = 0$. In addition, we have $\Phi'(\varpi) > 0$. From the definition of ϖ, we have

$$\frac{1}{\alpha}\Phi'(\varpi)\frac{\partial \varpi}{\partial \alpha} = -\frac{\beta_1\beta_2\varpi^{\beta_2 - 1}}{\alpha^2}\left((1 - b - \beta_1)\varpi^{\beta_1 - \beta_2} - (1 - b - \beta_2)\right).$$

Since $\varpi > 1$, we have $(1 - b - \beta_1)\varpi^{\beta_1 - \beta_2} - (1 - b - \beta_2) < -(\beta_1 - \beta_2) < 0$. Hence $\frac{\partial \varpi}{\partial \alpha} < 0$. The existence and uniqueness of X, Y, K_1 and K_2 follow. When $b = 1$, $Y = A$ and ϖ is defined by

$$\varpi^{\beta_1}(\beta_1 - \alpha(\beta_1 - 1)\varpi) = \varpi^{\beta_2}(\beta_2 - \alpha(\beta_2 - 1)\varpi). \blacksquare \tag{20}$$

A.6.2. Proof of Existence and Uniqueness of (X, Y, K_1, K_2) when $z_1^* = 0$

We want to show existence and uniqueness for the following 4 by 4 non-linear system:

$$\alpha X = A + K_1 X^{\beta_1} + K_2 X^{\beta_2} \tag{21}$$

$$A = (\beta_1 - 1)K_1 X^{\beta_1} + (\beta_2 - 1)K_2 X^{\beta_2} \tag{22}$$

$$Y = A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2} \tag{23}$$

$$A = (\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2}. \tag{24}$$

First of all, notice that K_1 and K_2 must have opposite sign otherwise the function

$$\Phi : y \mapsto (\beta_1 - 1)K_1 y^{\beta_1} + (\beta_2 - 1)K_2 y^{\beta_2},$$

is monotonic and therefore the equation $\Phi(y) = A$ cannot have two distinct roots. Then we have

$$\alpha X = \beta_1 K_1 X^{\beta_1} + \beta_2 K_2 X^{\beta_2},$$

and since $X > 0, \beta_1 > 0$ and $\beta_2 < 0$, it must be the case that $K_1 > 0$ and $K_2 < 0$. Then combining relationships (23) and (24) yields

$$\begin{aligned}
(\beta_1 - \beta_2)K_1 Y^{\beta_1} &= \beta_2 A - (\beta_2 - 1)Y \\
-(\beta_1 - \beta_2)K_2 Y^{\beta_2} &= \beta_1 A - (\beta_1 - 1)Y.
\end{aligned}$$

Once again, define $\varpi = \frac{X}{Y} \geq 1$, and we have

$$\begin{aligned}
(\beta_1 - 1)(\beta_2 A - (\beta_2 - 1)Y)\varpi^{\beta_1} - (\beta_2 - 1)(\beta_1 A - (\beta_1 - 1)Y)\varpi^{\beta_2} &= (\beta_1 - \beta_2)A \\
(\beta_2 A - (\beta_2 - 1)Y)\varpi^{\beta_1} - (\beta_1 A - (\beta_1 - 1)Y)\varpi^{\beta_2} &= (\beta_1 - \beta_2)(\alpha X - A).
\end{aligned}$$

Eliminating Y leads to

$$X = \frac{A\left((\beta_1 - \beta_2)\varpi^{\beta_1 + \beta_2} + \beta_1(\beta_2 - 1)\varpi^{\beta_1} - \beta_2(\beta_1 - 1)\varpi^{\beta_2}\right)}{\alpha(\beta_1 - 1)(\beta_2 - 1)(\varpi^{\beta_1} - \varpi^{\beta_2})},$$

and it follows that ϖ is implicitly defined by

$$\begin{aligned}
& -\alpha\beta_2(\beta_1 - 1)\varpi^{\beta_1} + \beta_1(\beta_2 - 1)\varpi^{\beta_1 - 1} + (\beta_1 - \beta_2)\varpi^{\beta_1 + \beta_2 - 1} \\
= \; & -\beta_2(\beta_1 - 1)\varpi^{\beta_2 - 1} + \alpha\beta_1(\beta_2 - 1)\varpi^{\beta_2} + \alpha(\beta_1 - \beta_2) = 0.
\end{aligned}$$

Define

$$\begin{aligned}
& [1, \infty) && \rightarrow && \mathbb{R} \\
\Phi : \; & y && \mapsto && -\alpha\beta_2(\beta_1 - 1)y^{\beta_1} + \beta_1(\beta_2 - 1)y^{\beta_1 - 1} + (\beta_1 - \beta_2)y^{\beta_1 + \beta_2 - 1} \\
& && && -\beta_2(\beta_1 - 1)y^{\beta_2 - 1} + \alpha\beta_1(\beta_2 - 1)y^{\beta_2} + \alpha(\beta_1 - \beta_2).
\end{aligned}$$

We want to show that there is a unique ϖ such that $\Phi(\varpi) = 0$. Φ is continuously differentiable and $\Phi(1) = 0$. Then

$$\begin{aligned}
\Phi'(y) &= -\alpha\beta_1\beta_2(\beta_1 - 1)y^{\beta_1 - 1} + \beta_1(\beta_2 - 1)(\beta_1 - 1)y^{\beta_1 - 2} + (\beta_1 - \beta_2)\beta_1 + \beta_2 - 1)y^{\beta_1 + \beta_2 - 2} \\
& \quad -\beta_2(\beta_1 - 1)(\beta_2 - 1)y^{\beta_2 - 2} + \alpha\beta_1\beta_2(\beta_2 - 1)y^{\beta_2 - 1}.
\end{aligned}$$

$\Phi'(1) = -\beta_1\beta_2(\beta_1-\beta_2)(\alpha-1) < 0$. In addition, $\Phi'(y)$ has the same sign as

$$-\alpha\beta_1\beta_2(\beta_1-1)y^{1-\beta_2} + \beta_1(\beta_2-1)(\beta_1-1)y^{-\beta_2} + (\beta_1-\beta_2)\beta_1 + \beta_2 - 1)$$
$$-\beta_2(\beta_1-1)(\beta_2-1)y^{-\beta_1} + \alpha\beta_1\beta_2(\beta_2-1)y^{1-\beta_1}.$$

Then define

$$\Theta: \begin{array}{lcl} [1,\infty) & \to & \mathbb{R} \\ y & \mapsto & \begin{array}{l} -\alpha\beta_1\beta_2(\beta_1-1)y^{1-\beta_2} + \beta_1(\beta_2-1)(\beta_1-1)y^{-\beta_2} + (\beta_1-\beta_2)\beta_1 + \beta_2 - 1) \\ -\beta_2(\beta_1-1)(\beta_2-1)y^{-\beta_1} + \alpha\beta_1\beta_2(\beta_2-1)y^{1-\beta_1}. \end{array} \end{array}$$

We know that $\Theta(1) = -\beta_1\beta_2(\beta_1-\beta_2)(\alpha-1) < 0$ and

$$\Theta'(x) = \alpha\beta_1\beta_2(\beta_1-1)(\beta_2-1)\left(\alpha y-1\right)y^{-\beta_1-1}(y^{\beta_1-\beta_2}-1) \geq 0.$$

Hence, Θ is strictly decreasing on $[1,\frac{1}{\alpha}]$ and strictly increasing on $[\frac{1}{\alpha},\infty)$. Since $\lim\limits_{\infty}\Theta = \infty$, we conclude that Θ has a unique root y^* in $(\frac{1}{\alpha},\infty)$. Thus, Φ is decreasing on $[1,y^*]$ and increasing on $[y^*,\infty)$ with $\lim\limits_{\infty}\Phi = \infty$. This shows that Φ has a unique root $\varpi > \frac{1}{\alpha}$ and $\Phi'(\varpi) > 0$. From the definition of ϖ, we have

$$\Phi'(\varpi)\frac{\partial\varpi}{\partial\alpha} = \beta_2(\beta_1-1)\varpi^{\beta_1} - \beta_1(\beta_2-1)\varpi^{\beta_2} - (\beta_1-\beta_2).$$

Since $x \mapsto \beta_2(\beta_1-1)x^{\beta_1} - \beta_1(\beta_2-1)x^{\beta_2} - (\beta_1-\beta_2)$ is decreasing and since $\varpi > 1$, we have $\beta_2(\beta_1-1)\varpi^{\beta_1} - \beta_1(\beta_2-1)\varpi^{\beta_2} - (\beta_1-\beta_2) < 0$. Hence $\frac{\partial\varpi}{\partial\alpha} < 0$. The existence and uniqueness of X, Y, K_1 and K_2 follow. ∎

A.7 Proof of Proposition 2

Case 1: $z_1^* > 0$ is optimal. When α increases $f(1)$ must decrease. Since $f'(1) = (1-b)f(1)$, we deduce that when $b < 1 (b > 1)$, $\frac{\partial Y}{\partial\alpha} < 0$ ($\frac{\partial Y}{\partial\alpha} > 0$). Hence $\frac{1}{Y-A}\frac{\partial Y}{\partial\alpha} > 0, b \neq 1$. When $b = 1$, then $Y = A$, so $\frac{\partial Y}{\partial\alpha} = 0$. Then

$$\begin{aligned} \frac{\partial K_1}{\partial\alpha}Y^{\beta_1+1} &= (-\beta_1 K_1 Y^{\beta_1} + \frac{\beta_2(1-b-\beta_1)Y}{(\beta_1-\beta_2)(b-1)})\frac{\partial Y}{\partial\alpha} \\ &= \frac{K_1 Y^{\beta_1}}{Y-A}\frac{\partial Y}{\partial\alpha}(\beta_1 A + (1-\beta_1)Y). \end{aligned}$$

Recall that

$$(\beta_2-\beta_1)K_2X^{\beta_2} = -(\beta_1-1)\alpha X + \beta_1 A > 0,$$

so $\alpha X < \frac{\beta_1 A}{\beta_1 - 1}$ and since $Y < \alpha X$, it follows that $Y < \frac{\beta_1 A}{\beta_1 - 1}$. Therefore $\frac{\partial K_1}{\partial \alpha} > 0$. From

$$u = A\left(f'(u)\right)^{-\frac{1}{b}} + K_1\left(f'(u)\right)^{\frac{\beta_1-1}{b}} + K_2\left(f'(u)\right)^{\frac{\beta_2-1}{b}},$$

it follows that

$$\begin{aligned} & \left(f'(u)\right)^{-1}\left(\frac{A}{b}\left(f'(u)\right)^{-\frac{1}{b}} - \frac{\beta_1-1}{b}K_1\left(f'(u)\right)^{\frac{\beta_1-1}{b}} - \frac{\beta_2-1}{b}K_2\left(f'(u)\right)^{\frac{\beta_2-1}{b}}\right)\frac{\partial f'(u)}{\partial \alpha} \\ = \ & \frac{\partial K_1}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_1-1}{b}} + \frac{\partial K_2}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_2-1}{b}}. \end{aligned}$$

The sign of the LHS is the same as $\frac{\partial K_1}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_1-\beta_2}{b}} + \frac{\partial K_2}{\partial \alpha}$ and $u \mapsto \frac{\partial K_1}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_1-\beta_2}{b}} + \frac{\partial K_2}{\partial \alpha}$ achieves its minimum at 1 and its maximum at $u = \alpha$. Then

$$\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} = Y^{-(1+\beta_2)}\frac{\partial Y}{\partial \alpha}(A - (\beta_1-1)K_1Y^{\beta_1} - (\beta_2-1)K_2Y^{\beta_2}).$$

Since $A - (\beta_1-1)K_1Y^{\beta_1} - (\beta_2-1)K_2Y^{\beta_2} > 0$, when $b > 1$, $\frac{\partial Y}{\partial \alpha} > 0$, so $\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} > 0$. In addition, when $b = 1$, $\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} = 0$. Hence, when $b \geq 1$, $\frac{\partial f'(u)}{\partial \alpha} > 0$. As $c^* = M\left(f'(u)\right)^{-\frac{1}{b}}$, if $b \geq 1$, $\frac{\partial c^*}{\partial \alpha} < 0$. Conversely, when $b < 1$, we have $\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} < 0$. Then

$$\begin{aligned} \frac{\partial K_1}{\partial \alpha}X^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} &= \frac{X^{-\beta_2}}{(Y-A)Y}\frac{\partial Y}{\partial \alpha}\Big(K_1X^{\beta_1}(Y(1-\beta_1)+\beta_1A) + K_2X^{\beta_2}(Y(1-\beta_2)+\beta_2A)\Big) \\ &= \frac{X^{-\beta_2}}{(Y-A)Y}\frac{\partial Y}{\partial \alpha}A(\alpha X - Y) > 0. \end{aligned}$$

Hence, there exists u^*_α in $(\alpha, 1)$, so that $\frac{\partial f'(u)}{\partial \alpha} > 0$ on $[\alpha, u^*_\alpha)$ and $\frac{\partial f'(u)}{\partial \alpha} < 0$ on $(u^*_\alpha, 1]$. We conclude that $\frac{\partial c^*}{\partial \alpha} < 0$ on $[\alpha, u^*_\alpha)$ and $\frac{\partial c^*}{\partial \alpha} > 0$ on $(u^*_\alpha, 1]$. ■

Case 2: $z_1^* = 0$ is optimal. In this case, we have

$$Y\frac{\partial Y}{\partial \alpha} = (\beta_1K_1Y^{\beta_1} + \beta_2K_2Y^{\beta_2})\frac{\partial Y}{\partial \alpha} + \frac{\partial K_1}{\partial \alpha}Y^{\beta_1+1} + \frac{\partial K_2}{\partial \alpha}Y^{\beta_2+1}.$$

Recall that $Y = \beta_1K_1Y^{\beta_1} + \beta_2K_2Y^{\beta_2}$, so

$$\frac{\partial K_1}{\partial \alpha}Y^{\beta_1} + \frac{\partial K_2}{\partial \alpha}Y^{\beta_2} = 0,$$

and therefore, $\frac{\partial K_1}{\partial \alpha}$ and $\frac{\partial K_2}{\partial \alpha}$ must have opposite signs. Similarly

$$X + \alpha\frac{\partial X}{\partial \alpha} = \frac{\partial K_1}{\partial \alpha}X^{\beta_1} + \frac{\partial K_2}{\partial \alpha}X^{\beta_2} + \frac{1}{X}(\beta_1K_1X^{\beta_1} + \beta_2K_2X^{\beta_2})\frac{\partial X}{\partial \alpha}.$$

Since $\alpha X = \beta_1K_1X^{\beta_1} + \beta_2K_2X^{\beta_2}$, we find that

$$\frac{\partial K_1}{\partial \alpha}X^{\beta_1} + \frac{\partial K_2}{\partial \alpha}X^{\beta_2} = X > 0.$$

As in case 1, to determine the sign of $\frac{\partial f'(u)}{\partial \alpha}$, we need to investigate the sign of $\frac{\partial K_1}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_1-\beta_2}{b}} + \frac{\partial K_2}{\partial \alpha}$. Set $y = (f'(u))^{\frac{1}{b}}$ and define

$$\begin{array}{rlcl} & [Y,X] & \to & \mathbb{R} \\ \Phi: & y & \mapsto & \frac{\partial K_1}{\partial \alpha} y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha}, \end{array}$$

Φ is a continuous and differentiable function with $\Phi(Y) = 0$. Then, F is monotonic and $\Phi(X) > 0$. Hence, it must be the case that $\frac{\partial K_1}{\partial \alpha} > 0$ and $\frac{\partial K_2}{\partial \alpha} < 0$ and Φ is positive on $[Y, X]$. We conclude that $\frac{\partial c^*}{\partial \alpha} < 0$. ■

A.8. Proof of Proposition 3

Note that

$$\frac{\partial}{\partial y}\left(\frac{z^*}{W}\right) = -\frac{(\mu - r)y^{\beta_1-1}}{b\sigma^2}\left(\frac{A(\beta_1^2 K_1 + \beta_2^2 K_2 y^{\beta_2-\beta_1}) + K_1 K_2(\beta_1-\beta_2)^2 y^{\beta_2}}{(A + K_1 y^{\beta_1} + K_2 y^{\beta_2})^2}\right).$$

Define the auxiliary function

$$\begin{array}{rlcl} & [Y,X] & \to & \mathbb{R} \\ \Psi: & y & \mapsto & A(\beta_1^2 K_1 + \beta_2^2 K_2 y^{\beta_2-\beta_1}) + K_1 K_2(\beta_1-\beta_2)^2 y^{\beta_2}. \end{array}$$

Ψ is strictly increasing so it has at most one root. Since when $u = 1$, we have $\frac{z^*}{W} \geq 0$, then it must be the case that either Ψ has no root and is strictly positive or Ψ has one root so it is first negative and then positive. We examine the sign of $\Psi(Y)$.

Case 1. When $z_1^* = 0$ is optimal, since $z_\alpha^* = z_1^* = 0$, it must be the case that Ψ indeed has a root. $\frac{z^*}{W}$ is hump-shaped in u. ■

Case 2. When $F_2(M, M) = 0$ is optimal, we have $\Psi(Y)$

$$\begin{aligned} \Psi(Y) &= A(\beta_1^2 K_1 + \beta_2^2 K_2 Y^{\beta_2-\beta_1}) + K_1 K_2(\beta_1-\beta_2)^2 Y^{\beta_2} \\ &= \frac{\beta_1(\beta_1-\beta_2)K_1}{\beta_1+b-1}\left(A(\beta_1+\beta_2+b-1) + (1-b-\beta_2)(1-b-\beta_1)\frac{Y-A}{b-1}\right) \\ &= \frac{\beta_1(\beta_1-\beta_2)K_1}{(1-b-\beta_2)(b-1)}\left(\frac{1}{\theta} - Y\right), \end{aligned}$$

since $-\frac{\beta_1\beta_2 A}{(\beta_1+b-1)(1-b-\beta_2)} = \frac{1}{\theta}$ as $\beta_1\beta_2 A = -\frac{1}{\frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2}$ and $(\beta_1+b-1)(1-b-\beta_2) = \frac{\theta}{\frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2}$. Hence $\Psi(Y)$ is positive exactly if and only if

$$Y < \frac{1}{\theta}\ (Y > \frac{1}{\theta}) \text{ whenever } b > 1\ (b < 1). \tag{25}$$

We can conclude that $\frac{z^*}{W}$ is strictly increasing in u exactly when relationship (25) is satisfied otherwise it is hump-shaped. A sufficient condition for Ψ to be always positive is $\theta < r$ ($\theta > r$) whenever $b > 1$ ($b < 1$).

Case $b = 1$. We have $Y = A$ and

$$\begin{aligned}\Psi(Y) &= K_1(\beta_1-\beta_2)(A(\beta_1+\beta_2-(\beta_1-\beta_2)K_1Y^{\beta_1}) \\ &= \frac{AK_1(\beta_1-\beta_2)}{\varpi^{\beta_1}}\Big((\beta_1+\beta_2)\varpi^{\beta_1}-(\beta_2-\alpha(\beta_2-1)\varpi)\Big),\end{aligned}$$

where ϖ is defined by relationship (20). $\frac{z^*}{W}$ is strictly increasing (hump shaped) in u exactly when $\Psi(Y) \geq 0 (\leq 0)$. ∎

A.9. Proof of Proposition 5

Let $y = f'(u)^{\frac{1}{b}}$. As seen before in this Appendix

$$\frac{1}{y}\frac{\partial y}{\partial \alpha} = \frac{y^{\beta_1}\frac{\partial K_1}{\partial \alpha}+y^{\beta_2}\frac{\partial K_2}{\partial \alpha}}{A-(\beta_1-1)K_1y^{\beta_1}-(\beta_2-1)K_2y^{\beta_2}}.$$

Since

$$z^* = \frac{\mu-r}{b\sigma^2}\Big(Ay^{-1}-(\beta_1-1)K_1y^{\beta_1-1}-(\beta_2-1)K_2y^{\beta_2-1}\Big).$$

It follows that

$$\begin{aligned}\frac{\partial z^*}{\partial \alpha} &= -\frac{\mu-r}{b\sigma^2 y}\left((\beta_1-1)\frac{\partial K_1}{\partial \alpha}y^{\beta_1}+(\beta_2-1)\frac{\partial K_2}{\partial \alpha}y^{\beta_2}+(A+(\beta_1-1)^2K_1y^{\beta_1}+(\beta_2-1)^2K_2y^{\beta_2}\frac{\partial y}{\partial \alpha}\right) \\ &= -\frac{(\mu-r)y^{\beta_1+\beta_2}}{b\sigma^2 y}\frac{A(\beta_1\frac{\partial K_1}{\partial \alpha}y^{-\beta_2}+\beta_2\frac{\partial K_2}{\partial \alpha}y^{-\beta_1})+(\beta_1-\beta_2)((\beta_1-1)K_1\frac{\partial K_2}{\partial \alpha}-(\beta_2-1)K_2\frac{\partial K_2}{\partial \alpha}}{A-(\beta_1-1)K_1y^{\beta_1}-(\beta_2-1)K_2y^{\beta_2}}.\end{aligned}$$

The denominator of the above fraction is positive. In order to investigate the sign of its numerator, let us define an auxiliary function

$$\begin{aligned}&[Y,X] \;\rightarrow\; \mathbb{R} \\ \Theta:\;& y \;\mapsto\; A(\beta_1\tfrac{\partial K_1}{\partial \alpha}y^{-\beta_2}+\beta_2\tfrac{\partial K_2}{\partial \alpha}y^{-\beta_1})+(\beta_1-\beta_2)((\beta_1-1)K_1\tfrac{\partial K_2}{\partial \alpha}-(\beta_2-1)K_2\tfrac{\partial K_2}{\partial \alpha}).\end{aligned}$$

Θ is continuous and differentiable and

$$\Theta'(y) = -\beta_1\beta_2A(\frac{\partial K_1}{\partial \alpha}y^{-\beta_2-1}+\frac{\partial K_2}{\partial \alpha}y^{-\beta_1-1}).$$

Since $\frac{\partial K_1}{\partial \alpha} > 0$ and $\frac{\partial K_2}{\partial \alpha} < 0$, Θ' is strictly increasing and either $\Theta(Y) \geq 0$ or F achieves its minimum at y^* such that

$$\frac{\partial K_1}{\partial \alpha}(y^*)^{\beta_1}+\frac{\partial K_2}{\partial \alpha}(y^*)^{\beta_2} = 0.$$

If Θ achieves its minimum at y^*, then $\Theta(y^*) = (\beta_1 - \beta_2)(y^*)^{-\beta_2} \frac{\partial K_1}{\partial \alpha} > 0$, and in this case, Θ is positive on $[Y, X]$. Otherwise, Θ' is positive and Θ is strictly increasing. To prove that Θ is positive on $[Y, X]$, it is enough to show that $\Theta(Y) \geq 0$ or equivalently that z_1^* is a decreasing function of α. There are two cases.

Case 1: $F_2(M, M) = 0$ is optimal. In this case

$$z_1^* = \frac{\mu - r}{b\sigma^2 Y} \left(A - (\beta_1 - 1) K_1 Y^{\beta_1} - (\beta_2 - 1) K_2 Y^{\beta_2} \right).$$

If $b = 1$, then $Y = A$ and

$$\frac{\partial z_1^*}{\partial \alpha} = -\frac{\mu - r}{b\sigma^2 A} \left((\beta_1 - 1) \frac{\partial K_1}{\partial \alpha} A^{\beta_1} + (\beta_2 - 1) \frac{\partial K_2}{\partial \alpha} A^{\beta_2} \right) < 0.$$

If $b \neq 1$, we have

$$z_1^* = \frac{\mu - r}{b\sigma^2} \left(1 + \frac{\beta_1 \beta_2 (Y - A)}{(b - 1) Y} \right).$$

Therefore

$$\frac{\partial z_1^*}{\partial \alpha} = \frac{\mu - r}{b\sigma^2} \left(-\frac{\beta_1 \beta_2 A}{Y^2} \frac{\frac{\partial Y}{\partial \alpha}}{1 - b} \right).$$

Since $1 - b$ and $\frac{\partial Y}{\partial \alpha}$ have opposite signs, we conclude that $\frac{\partial z_1^*}{\partial \alpha} < 0$. ■

Case 2: $z_1^* = 0$ is optimal. In this case, z_1^* is independent of α, so $\frac{\partial z_1^*}{\partial \alpha} = 0$ and $\Theta(Y) = 0$. The proof is complete. ■

A.10 Representation of $\frac{c^*}{M}$, c^* and W as Stochastic Processes

Process $\frac{c^*}{M}$. For u in $(\alpha, 1)$, recall that $u = G(\frac{c^*}{M})$ so denoting $H = G^{-1}$ we have

$$H'(u) = \frac{1}{G'(\frac{c^*}{M})} \text{ and } H''(u) = -\frac{G''(\frac{c^*}{M})}{(G'(\frac{c^*}{M}))^3}.$$

Applying Ito lemma, we find

$$\begin{aligned} d\left(\frac{c_t^*}{M_t}\right) &= H'(u_t) du_t + \frac{\sigma^2}{2} \left(\frac{z_t}{M_t}\right)^2 H''(u_t) dt \\ &= \frac{rG(\frac{c_t^*}{M_t}) - \frac{c_t^*}{M_t} + \frac{(\mu-r)^2}{b\sigma^2} c_t^* G'(\frac{c_t^*}{M_t}) - \frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2 \left(\frac{c_t^*}{M_t}\right)^2 G''(\frac{c_t^*}{M_t})}{G'(\frac{c_t^*}{M_t})} dt \\ &\quad + \frac{\mu - r}{b\sigma} \frac{c_t^*}{M_t} dw_t. \end{aligned}$$

Then

$$\begin{aligned}
& rG(\frac{c_t^*}{M_t}) - \frac{c_t^*}{M_t} - \frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2 \left(\frac{c_t^*}{M_t}\right)^2 G''(\frac{c_t^*}{M_t}) \\
= \quad & (r - \frac{1}{A})\left((r - \frac{\beta_1(\beta_1 - 1)}{2}\left(\frac{\mu - r}{b\sigma}\right)^2)K_1\left(\frac{c_t^*}{M_t}\right)^{1-\beta_1}\right. \\
& \left. + (r - \frac{\beta_2(\beta_2 - 1)}{2}\left(\frac{\mu - r}{b\sigma}\right)^2)K_2\left(\frac{c_t^*}{M_t}\right)^{1-\beta_2} + A\frac{c_t^*}{M_t}\right) \\
= \quad & (r - \frac{1}{A})\frac{c_t^*}{M_t}G'(\frac{c_t^*}{M_t}).
\end{aligned}$$

So

$$d\left(\frac{c_t^*}{M_t}\right) = \frac{c_t^*}{M_t}\left((r - \frac{1}{A} + \frac{(\mu - r)^2}{b\sigma^2})dt + \frac{\mu - r}{b\sigma}dw_t\right).$$

For u in $[0,1]$, we have $\frac{1}{X} \leq \frac{c^*}{M} \leq \frac{1}{Y}$. Define the geometric Brownian motion v such that $v_0 = X(f'(\frac{W_0}{M_0}))^{-\frac{1}{b}}$ and

$$dv_s = v_s\left((r - \frac{1}{A} + \frac{(\mu - r)^2}{b\sigma^2})ds + \frac{\mu - r}{b\sigma}dw_s\right).$$

Then, a representation of the process $\frac{c^*}{M}$ is

$$\frac{c_t^*}{M_t} = \frac{v_t e^{L_t - U_t}}{X},$$

where the processes L and U are increasing and continuous with $L_0 = U_0 = 0$ and

$$\begin{aligned}
L_t &= \sup_{0 \leq s \leq t}[\log v_s - U_s]^- \\
U_t &= \sup_{0 \leq s \leq t}\left[\log\frac{X}{Y} - \log v_s - L_s\right]^-.
\end{aligned}$$

See Harrison (1985) p 22. ∎

Consumption and Wealth Processes. The wealth process is given by $W_t = M_t G(\frac{c_t^*}{M_t})$ and let $H = G^{-1}$ so that $c_t^* = M_t H(\frac{W_t}{M_t})$.

Case 1: $F_2(M, M) = 0$ is optimal. Using Ito lemma for semimartingales, we find that

$$\log H\left(\frac{W_t}{M_t}\right) = \log H\left(\frac{W_0}{M_0}\right) + (\frac{r - \theta}{b} + \frac{(\mu - r)^2}{2b\sigma^2})t + \frac{\mu - r}{b\sigma}w_t - \frac{H'(1)}{H(1)}\log\frac{M_t}{M_0},$$

Note that $\frac{H'(1)}{H(1)} = \frac{Y}{G'(\frac{1}{Y})} > 0$. As explained in Grossman and Zhou (1993), the quantity $\log H\left(\frac{W_t}{M_t}\right)$ is bounded from above by $\log H(1) = -\log Y$ and $\frac{H'(1)}{H(1)}\log\left(\frac{M_t}{M_0}\right) > 0$ serves as a regulator to keep the arithmetic Brownian motion from exceeding $-\log Y$. Define

$$l_t = \sup_{0 \leq s \leq t}\left[\log H\left(\frac{W_0}{M_0}\right) + (\frac{r - \theta}{b} + \frac{(\mu - r)^2}{2b\sigma^2})s + \frac{\mu - r}{b\sigma}w_s + \log Y\right]^+.$$

It follows that $M_t = M_0 e^{\frac{Y}{G'(\frac{1}{Y})} l_t}$ and a representation of the wealth process and consumption process is

$$\begin{aligned} W_t &= M_0 e^{\frac{Y}{G'(\frac{1}{Y})} l_t} G\left(H\left(\frac{W_0}{M_0}\right) e^{(\frac{r-\theta}{b} + \frac{(\mu-r)^2}{2b\sigma^2})t + \frac{\mu-r}{b\sigma} w_t - l_t} \right) \\ c_t^* &= H\left(\frac{W_0}{M_0}\right) e^{(\frac{r-\theta}{b} + \frac{(\mu-r)^2}{2b\sigma^2})t + \frac{\mu-r}{b\sigma} w_t - l_t}. \end{aligned}$$

Case 2: $z_1^* = 0$ is optimal. When wealth hits its maximum to date for the first time τ_0, we have

$$dW_t = (r - \frac{1}{Y}) W_t dt.$$

Upper Absorbing Barrier. If $\frac{1}{Y} < r$, for all $t \geq \tau_0$, $W_t = M_t$, $c_t^* = \frac{M_t}{Y}$ and $z_t = 0$. A representation of the wealth and consumption is

$$\begin{aligned} W_t &= \begin{cases} M_0 G\left(H\left(\frac{W_0}{M_0}\right) e^{(\frac{r-\theta}{b} + \frac{(\mu-r)^2}{2b\sigma^2})t + \frac{\mu-r}{b\sigma} w_t} \right), \text{ for } t \leq \tau_0 \\ M_0 e^{(r-\frac{1}{Y})(t-\tau_0)}, \text{ for } t \geq \tau_0 \end{cases} \\ c_t^* &= \begin{cases} H(\frac{W_0}{M_0}) e^{(\frac{r-\theta}{b} + \frac{(\mu-r)^2}{2b\sigma^2})t + \frac{\mu-r}{b\sigma} w_t}, \text{ for } t \leq \tau_0 \\ \frac{M_0 e^{(r-\frac{1}{Y})(t-\tau_0)}}{Y}, \text{ for } t \geq \tau_0. \end{cases} \end{aligned}$$

Upper Reflecting Barrier. If $r \leq \frac{1}{Y}$, wealth is driven down immediately after hitting its peak and cannot exceed M_0. Consumption and wealth processes are given by

$$\begin{aligned} c_t^* &= \frac{M_0}{X} v_t e^{L_t - U_t} \\ W_t &= M_0 G(\frac{v_t e^{L_t - U_t}}{X}). \blacksquare \end{aligned}$$

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