Speculative Risk Averse Investor Behavior in a Pure Exchange Economy

Hervé Roche*
Centro de Investigación Económica
Instituto Tecnológico Autónomo de México
Av. Camino a Santa Teresa No 930
Col. Héroes de Padierna
10700 México, D.F.
E-mail: hroche@itam.mx

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Abstract
Within a dynamic general equilibrium we provide conditions on agents’ preferences and expectations under which a speculation phenomenon as related in Harrison and Kreps (1978) arises. Namely, whatever beliefs disagreements, when investors are more risk averse than myopic agents, no speculation phenomenon occurs. When agents are less risk averse than myopic agents, the equilibrium asset price can exceed even the more optimistic investor’s valuation and does so if agents are risk neutral enough. More participants in the market can enhance the speculation phenomenon. Conversely, when agents are extremely risk averse, we may have a bear market in which the equilibrium asset price can drop below even the most pessimistic market fundamental value. The case of heterogeneous preferences is also examined.

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1. Introduction

This idea of heterogeneous beliefs across agents is not new in the financial economics literature\(^1\). However, it is only in the recent years\(^2\) that heterogeneous beliefs became more wildly accepted and there has been a surge of interest from scholars who have tried to use agents’ disagreement about some non-observables in order to explain market over-reaction or some financial puzzles\(^3\). In particular, beliefs dispersion has been a key ingredient to analyze the speculation phenomenon. According to Kaldor (1939), “Speculation [...] may be defined as the purchase (or sale) of goods with the view to resale (repurchase) at a later date, where the motive behind such action is the expectation of a change in the relevant prices relatively to the ruling price and not a gain accruing through use, or any kind of transformation effected in them, or transfer between markets.”

Speculation arises when the right to resell an asset makes investors willing to pay more for it that they would have pay if obliged to hold it for ever (market fundamental value). One of the workhorse model on speculation is the paper by Harrison and Kreps (1978) to which we refer in the sequel as the HK model. They consider a discrete time economy in which two risk neutral investors with infinite wealth trade one stock whose dividend can take only two values 0 or 1. Short sales are forbidden and although agents have access to the same information, they have heterogeneous subjective expectations about the transition probabilities of the dividend. One agent thinks that being in state 0, it is more likely to switch to state 1 where a dividend is paid. Alternatively, in state 1, the second agent is more optimistic about staying in this state and enjoying a dividend. This opens up the possibility of expected capital gains leading to a speculation phenomenon. At the equilibrium, one agent would like to hold all the shares in one state in order to sell them to the other agent at a price strictly greater than the fundamental value as soon as a switch of state occurs. The second agent is willing to buy all the shares because she anticipates that the first agent would like to hold them back when the state of the economy switch back to its original state and so on. Morris (1996) introduces learning in the HK model by assuming that traders ignore the true probability of receiving a dividend next period. Initially, they have heterogeneous prior beliefs about this probability and observing the realizations of the dividend process allows them to update their beliefs in a Bayesian fashion. Ultimately, all the beliefs will converge and speculation will dampen. The author relates this issue to

\(^1\)Rubinstein (1974) examines conditions, included some on heterogeneous beliefs, under which aggregation is possible. Williams (1977) investigates the impact of heterogeneous beliefs on the traditional Capital Asset Pricing Model. Feiger (1976) explores the idea of buying to resell generated heterogeneous expectations. Jarrow (1980) studies a two period model with CARA investors maximizing their utility of terminal wealth under normal returns in presence of short sales constraints. He points out that a substitution effect between securities due to returns correlation along with short sales constraints can lead to an increase or decrease of the equilibrium asset prices. This result contrasts with Miller (1977) who asserts that short sales always enhance the equilibrium asset prices.

\(^2\)See Morris (1995) for an interesting discussion on a departure from the common prior assumption in economics.

\(^3\)For instance, see Abel (1989) for an analysis of the equity premium under heterogeneous beliefs and more recently David (2004)).
initial public offerings (IPO) for which initial prices appears to be too high with respect to their long run values. He concludes that small differences in beliefs can generate a large speculative premium.

Both articles characterize the speculation phenomenon for risk neutral investors within a partial equilibrium framework. In this paper, we consider a Lucas tree version of the HK model similar to Abel’s (1989). Time is discrete and infinite. There is a single tree and two states of nature. There is some incomplete information is about the transition probabilities governing regime changes. Risk averse investors with CRRA preferences trade a risky asset and a riskless bond whose prices are endogenously determined. Our main focus is to investigate under which conditions on preferences and beliefs a speculative phenomenon arises (or does not) within a general equilibrium framework.

1.1. Related Literature

An excellent survey of the effects of heterogeneous beliefs on trading and speculation is provided in Scheinkman and Xiong (2004).

Extending some earlier work by Lintner (1969), Varian (1989) investigates the impact of heterogeneous beliefs on trading volume and asset price. He obtains that a large beliefs dispersion generates high trade among agents and can lower or increase the equilibrium asset price depending on the curvature of agents’ utility function. Harris and Raviv (1993), Kandel and Pearson (1995) and Odean (1995) focus on trading volume when investors have heterogeneous beliefs. Harrison and Stein (2003) develop a discrete time model in which heterogeneous beliefs and short sales constraints can lead to market crashes. Blume and Easley (2002) investigate conditions on discount rate factors and beliefs for traders to survive in complete and incomplete markets.

A large class of recent models are nested within a continuous-time framework with a Brownian motion uncertainty structure in which agents have incomplete information about some process (for instance the average dividend growth rate) and upon receiving some information from signals, update their beliefs according to Bayes rules. For instance, Kogan, Ross, Wang and Westerfield (2004) consider a general equilibrium a la Lucas with a Brownian motion uncertainty structure and are concerned with the survival of irrational traders. Building on the work of Detemple (1986) and Gennette (1986), Detemple and Murthy (1994), Feldman (1992) and Zapatero (1998) consider a continuous-time equilibrium framework where investors have logarithmic utility preferences and learn about the true value of the average growth rate of aggregate production. Due to the specific feature of logarithmic preferences (myopia), the equilibrium asset price and the interest rate is a weighted average that oscillates between the most pessimistic and the most optimistic agent’s valuations. In Basak (2000), agents have incomplete information about an extraneous process that is believed to affect the pure exchange economy. Consumption plans turn out to be more volatile than in absence of non-fundamental uncertainty; the effect on the interest and risk premium depends on the properties of the absolute risk aversion of the agents’ utility functions. In Scheinkman and Xiong (2003), two groups of traders place different weights on informative
signals they receive about the fundamental value of the dividend. Short sales provides the
owner of a stock an option to resale the asset in the future to agents with more optimistic
CARA investors disagree about the precision of the terminal payoff of the asset and revise
their (normally distributed) beliefs upon receiving a public signal at each period. The net
position on a trader depends on whether her own estimation of the precision is larger or
smaller than the average precision. Trading volume appears to be proportional to the de-
gree of dispersion of beliefs. In Kyle and Lin (2002), CARA preference traders (agree to )
disagree about the importance of signals about the mean of the dividend growth rate and
investors allocate more weight to the signal they believe to be the more relevant. Aggregate
optimism leads to higher prices (and lower returns); aggregate overconfidence generates
excess volatility and mean reversion in prices, with low returns when the average signal is
high and vice versa. Finally, trading volume is proportional to how confident traders are.

Another class of papers combined public and private information revealed to agents
across time. Wang (1993) develops a dynamic general equilibrium model where informed
and uninformed investors about the dividend process trade a riskless and a risky asset. He
explores the impact of asymmetric information on equilibrium prices, price volatility,
risk premium, serial correlation in returns and optimal strategies. In He and Wang (1995),
CARA investors trade a riskless and a risky asset based on their private information and
on the public information included the one revealed by the equilibrium price of the stock.
In particular, they obtain that public information generates trading in the current period
with small price changes whereas private information induces trading in the current but
also future periods with large price changes. Allen, Morris and Shin (2002) develop an asset
pricing model under incomplete information in which heterogeneous investors receive both
private and public signals. Agents are aware of their own beliefs but do not know the other
agents’ beliefs. They realize the importance of the average expectations about returns so
that the problem can be seen as beauty contest set up as described by Keynes (1936).
Agents tend to put a high weight on public information leading to over-reaction in financial
markets.

Heterogeneous beliefs can also arise from private information. However, when agents are
rational and share common priors, Milgrom-Stokey (1982) and Tirole (1982) show that no-
trade is indeed an equilibrium and therefore private information cannot generate speculative
trading. Finally, other related works are the ones by Dumas (1989) and Wang (1996)
although the aim of these two papers is not to focus on speculation but rather on the impact
of preference heterogeneity on the equilibrium allocations in a pure exchange economy.

1.2. Results

We show the existence of an equilibrium and establish a necessary but not sufficient condi-
tion to experience speculation, namely that the CRRA coefficient of investor must less than
unity. When investors are close to risk neutrality, speculation always takes place provided
that agents have heterogeneous beliefs. Conversely, when agents are extremely risk averse,
we may have a bear market in which the equilibrium asset price can drop below even the
most pessimistic market fundamental value. More participants in the market can enhance the speculation phenomenon. When agents can learn about the transition probabilities upon the arrival of regime switches, we obtain similar conditions as in the fixed expectation case for speculation to arise.

The paper is organized as follows. Section 2 describes the economic setting and characterizes the equilibrium allocations when agents have fixed heterogeneous beliefs. Section 3 discusses the conditions under which speculation can occur or alternatively when the equilibrium asset price is below the more pessimistic agent’s fundamental value. Section 4 presents some extensions of the model. In section 5, investors update their beliefs about the transition probabilities upon the arrival of state switches. Section 6 concludes. Proofs of all results are collected in the appendix.

2. The economic setting

We consider an infinite horizon pure exchange economy a la Lucas (1978) in which two investors have different expectations about the law of motion of the dividend process $y$. As in Varian (1985), agents behave as passive expected utility maximizers with fixed beliefs. In section 5, this assumption is relaxed and agents revise their beliefs upon the arrival of new public information.

Information structure. There are only two states of world $i \in \{1, 2\}$. Between time $t$ and $t+1$, the change of states is governed by the following transition matrix

$$
\begin{array}{c|cc}
\text{time } t & \text{time } t+1 \\
\hline
y_1 & 1 - \mu_1 & \mu_1 \\
y_2 & \mu_2 & 1 - \mu_2 \\
\end{array}
$$

with $(\mu_1, \mu_2) \in (0,1)^2$. As proved in appendix 1, the conditional probability $P_{ij}(t) = P(y(t) = y_j | y(0) = y_i)$ is given by

$$P_{ii}(t) = \frac{\mu_j + \mu_i (1 - \mu_i - \mu_j)^t}{\mu_i + \mu_j},$$

(2.1)

for $i = j$ and $P_{ij}(t) = 1 - P_{ii}(t)$ for $i \neq j$. Agents have different assessments regarding the probabilities $\mu_1$ and $\mu_2$. Agent $k$ believes that $(\mu_1, \mu_2) = (\phi_{1k}, \phi_{2k})$ for $k = a, b$. The corresponding conditional probabilities are denoted by $p_{ijk}$ for agent $k$, for $k = a, b$ and $(i,j) \in \{1,2\}^2$. In addition, we make the following assumption.

Assumption A.1. Agents agree to disagree

As in the HK model, the two classes of investors have different beliefs and each class thinks its own view (model) of the world is correct. When learning is introduced in section, our framework is essentially the same as in Harris and Raviv (1993) where investors receive common information but differ in the way they interpret it, each investor being convinced
of the validity of her interpretation. Harris and Raviv (1993) refer to this as “differences of opinion”. These differences in beliefs cannot be interpreted as differences in information otherwise a no trade theorem as highlighted in Milgrom and Stokey (1982) would apply. We now examine some of the beliefs properties.

Some properties of the conditional probabilities

The parameter \( \mu_i \) measures the persistence of the state \( i \): The lower the parameter \( \mu_i \) the more persistent is state \( i \). The expected duration of regime \( i \) is \( \frac{\mu_j}{\mu_i + \mu_j} \). Given the analytical expressions in relationship (2.1), it can be shown that the functions \( p_{iia} \) and \( p_{iib} \) can cross each other at most once.

**P1: No crossing condition.** A necessary and sufficient condition \( p_{iia}(t) \geq p_{iib}(t) \) for all \( t \) in \( \mathbb{R}_+ \) is

\[
\phi_{ia} \leq \phi_{ib} \quad \phi_{ja} \geq \phi_{jb} \quad \phi_{ia} > \phi_{ib} .
\]

Note that we cannot have at the same time \( p_{iia}(t) \geq p_{iib}(t) \) and \( p_{jja}(t) \geq p_{jjb}(t) \) for all \( t \) in \( \mathbb{R}_+ \) unless \( p_a \equiv p_b \). Moreover, we have \( p_{iia}(t) \geq p_{iib}(t) \) and \( p_{jja}(t) \geq p_{jjb}(t) \) for all \( t \) in \( \mathbb{R}_+ \) exactly when \( \phi_{ia} \leq \phi_{ib} \) and \( \phi_{ia} \phi_{ja} = \phi_{ib} \phi_{jb} \).

**P2: Crossing condition.** A necessary and sufficient condition for \( p_{iia} \) and \( p_{iib} \) to cross is

\[
(\phi_{ia} - \phi_{ib})(\frac{\phi_{ja}}{\phi_{ja}} - \frac{\phi_{ia}}{\phi_{ia}}) > 0 .
\]

If \( \phi_{ia} \leq \phi_{ib} \) and \( \frac{\phi_{ja}}{\phi_{ia}} \leq \frac{\phi_{ja}}{\phi_{ib}} \), \( p_{iia} \) starts dominating \( p_{iib} \) and after some time the opposite occurs. It is easy to verify that \( p_{jja} \) always dominates \( p_{jjb} \).

In the sequel, \( E^I_{k,t} \) denotes the conditional expectation at time \( t \) according to agent \( k \)'s beliefs, being in state \( i \).

**Individual preferences.** There is a single perishable good available for consumption, the numéraire. Preferences of agent \( k \) are represented by a time additive utility function

\[
U_k(c_k, t) = E^I_{k,t} \left[ \sum_{s=t}^{\infty} \beta^{s-t} u_k(c_k(s)) \right] ,
\]

where the instantaneous utility function \( u_k \) of agent \( k \) \( (k = a, b) \) is twice continuously differentiable, increasing and strictly concave and \( 0 < \beta < 1 \) is the time discount factor. In addition, \( u_k \) satisfies the following Inada conditions: \( \lim_{c \to 0^+} u'_k(c) = \infty \) and \( \lim_{c \to \infty} u'_k(c) = 0 \). In the sequel, we assume that agents have identical CRRA utility functions

\[
u_k(c) = \begin{cases} \frac{c^{\gamma-1}}{1-\gamma} , & \gamma \neq 1 \\ \log c, & \gamma = 1 , \end{cases}
\]
where $\gamma > 0$ is the coefficient of constant relative risk aversion.

**The financial market.**

The single consumption good is produced by a unique tree delivering a fruit $y_i$ in state $i$, with $y_1 < y_2$. The financial market consists of a stock market where shares of the tree are traded. Let $z_k$ denote the number of shares of the tree owned by agent $k$ and $S$ denotes the price of the tree. The total number of shares is normalized to 1. In addition, there is a “money market” in which a locally risk-free security can be traded, i.e. investors can borrow or lend to each other without default at some endogenously determined interest rate $r$. The wealth $W_k$ of agent $k$ is the sum of the value $x_k$ invested into cash and the value $z_kS$ invested into the risky asset, i.e. $W_k = x_k + z_kS$. Being initially at time 0 in state $i$, agent $k$ is initially endowed with $z_{ik}$ shares so her initial wealth is $W_{ik}(0) = z_{ik}S_0 > 0$ and $x_{ik} = 0$, for $k \in \{a, b\}$.

**Feasibility.** A consumption plan $c_k$ for agent $k$ is feasible if there is a trading strategy $(x_k, z_k)$ financing it such that

$$W_k(t + 1) - W_k(t) = x_k(t)r(t) - c_k(t) + z_k(t)(y(t) + S(t + 1) - S(t))$$

$$W_k(t) \geq -W,$$

for some $W > 0$. The lower bound placed on the wealth is enough to rule out arbitrage opportunities.

**2.1. The agent problem**

Being in state $i$, agent $k$ maximizes her lifetime utility

$$\max_{(c_k, (x_k, z_k))} \mathbb{E}^i_{k,0} \left[ \sum_{s=0}^{\infty} \beta^s u_k(c_k(s)) \right]$$

$$W_k(t + 1) - W_k(t) = x_k(t)r(t) - c_k(t) + z_k(t)(y(t) + S(t + 1) - S(t))$$

$$W_k(t) \geq -W, \; z_{ik} > 0 \text{ given.}$$

At time $t$, let $J^i_k(W_k(t), t)$ be the value function for agent $k$ in state $i$. The non-Ponzi game or transversality condition for this problem can be written:

$$\lim_{T \to \infty} \mathbb{E}^i_{k,t} \left[ J^i_k(W_k(t + T), t + T) \right] = 0.$$  \hspace{1cm} (TC)

for all $k \in \{a, b\}$ and $i \in \{1, 2\}$.


In the HK model, two risk neutral investors (with an infinite wealth and no short sales) trade a unique stock whose ownership entitle them to receive some dividend $y$. Due to the
stationary nature of the problem, price $S_i$ in state $i$ is time independent and satisfies the recursive formulation

$$S_i = \max_{k \in \{a,b\}} E_{i,0}^k \left[ \sum_{s=1}^{\tau_{ij}} \beta^s y_i + \beta^{\tau_{ij}} S_j \right],$$

where $\tau_{ij}$ is the first (stopping) time a switch from state $i$ into state $j$ occurs. The Hamilton Jacobi Bellman (HJB) equation is given by

$$S_i = \max_{k \in \{a,b\}} \{ (1 - \phi_{ik}) (S_i + y_i) + \phi_{ik} (S_j + y_j) \}.$$

Following Harrison and Kreps (1978), we assume that $\phi_{1a} > \phi_{1b}$ and $\phi_{2a} > \phi_{2b}$ and conjecture that $S_2 + y_2 > S_1 + y_1$ (to be verified in the sequel). This implies that

$$S_1 = \beta \{ (1 - \phi_{1a}) (S_1 + y_1) + \phi_{1a} (S_2 + y_2) \}$$
$$S_2 = \beta \{ (1 - \phi_{2b}) (S_2 + y_2) + \phi_{2b} (S_1 + y_1) \}.$$

Solving the system leads to

$$S_1 = \frac{(\theta(1 - \phi_{1a}) + \phi_{2b}) y_1 + (1 + \theta) \phi_{1a} y_2}{\theta(\theta + \phi_{1a} + \phi_{2b})},$$
$$S_2 = \frac{(1 + \theta) \phi_{2b} y_1 + (\theta(1 - \phi_{2b}) + \phi_{1a}) y_2}{\theta(\theta + \phi_{1a} + \phi_{2b})},$$

with $\theta = \frac{1-\beta}{\beta}$. Since $y_2 > y_1$, it is easy to check that indeed $S_2 + y_2 > S_1 + y_1$ so the conjecture was correct. If agent $k$ were alone and holding all the stock, the prevailing equilibrium price would be

$$S_{ik} = \frac{(\theta(1 - \phi_{ik}) + \phi_{jk}) y_i + (1 + \theta) \phi_{ik} y_j}{\theta(\theta + \phi_{ik} + \phi_{jk})}.$$

In appendix 1, we show that indeed the conjecture is correct and that

$$S_i > \max \{ S_{ia}, S_{ib} \} \text{ for } i \in \{1,2\},$$

So indeed, speculation occurs in both states.

Our framework differs from the HK model since we consider a general equilibrium model in which the interest rate is not an exogenous constant $\theta$ but instead is endogenously determined and state dependent.

2.3. Equilibrium analysis

An equilibrium for this economy consists of

1. A triplet of optimal allocations $(c_k, z_k, x_k)$ that are solutions of agent $k$’s problem (P),
2. A price of the tree $S$ and an interest rate $r$

3. Market clearing conditions, for $i \in \{1, 2\}$.

\[
\begin{align*}
     c_{ia} + c_{ib} &= y_i \text{ (goods market)} \\
     z_{ia} + z_{ib} &= 1 \text{ (equity market)} \\
     x_{ia} + x_{ib} &= 0 \text{ (bond market)}. 
\end{align*}
\]

Due to Walras’ law, the bond market clearing condition is automatically satisfied when the two other markets clearing conditions are met. In the sequel, $c_{ijk}(s)$ denotes agent $k$’s planned consumption in state $j$, decided at time 0 in state $i$. We start with the benchmark case of an economy populated by homogenous beliefs and preferences individuals.

### 2.3.1. Homogenous investors

In this case, for $i = 1, 2$, the optimal allocations are

\[
c_{ik} = y_i, z_{ik} = 1 \text{ and } x_{ik} = 0.
\]

The equilibrium price of the tree $S_{ik}$ is given by

\[
S_{ik} = \frac{1}{u'(y_i)} \sum_{s=1}^{\infty} \beta^s u'_k(y(s))y(s) \quad \frac{\theta(1 - \phi_{ik}) + \phi_{jk})y_i + (1 + \theta)\phi_{ik} \left(\frac{y_i}{y_j}\right)^\gamma y_j}{\theta(\theta + \phi_{ik} + \phi_{jk})}.
\]

The following table provides some conditions under which $S_{ia} \geq S_{ib}$, for $i = 1, 2$.

<table>
<thead>
<tr>
<th>State</th>
<th>$\phi_{1a} \phi_{1b} &lt; \phi_{2a} \phi_{2b}$</th>
<th>$S_{1a} = S_{1b}$</th>
<th>$\phi_{1a} \phi_{2a} &gt; \phi_{2b} \phi_{1b}$</th>
<th>$S_{2a} = S_{2b}$</th>
<th>$\phi_{2a} \phi_{1a} &lt; \phi_{2b} \phi_{1b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>$\frac{\phi_{1a}}{\phi_{1b}} &lt; \frac{\phi_{2a}}{\phi_{2b}}$</td>
<td>$S_{1a} = S_{1b}$</td>
<td>$\frac{\phi_{1a}}{\phi_{2a}} &gt; \frac{\phi_{1b}}{\phi_{2b}}$</td>
<td>$S_{2a} = S_{2b}$</td>
<td>$\frac{\phi_{2a}}{\phi_{1b}} &lt; \frac{\phi_{2a}}{\phi_{2b}}$</td>
</tr>
</tbody>
</table>

The equilibrium interest rate $r_{ik}$ is such that

\[
\frac{1}{1 + r_i} u'(y_i) = \beta E^i_{k,t} \left[ u'(y(t+1)) \right],
\]

i.e.

\[
\frac{1}{1 + r_i} = \beta \left[ (1 - \phi_{ik}) + \phi_{ik} \left( \frac{y_i}{y_j} \right)^\gamma \right].
\]

Conditions for $r_{ia} \geq r_{ib}$ are

\[
\begin{align*}
     r_{1a} > r_{1b} & \iff \phi_{1a} > \phi_{1b} \\
     r_{2a} > r_{2b} & \iff \phi_{2a} < \phi_{2b}.
\end{align*}
\]

In any case, $\lim_{\gamma \to 0} r_{ik} = \theta$. 

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2.3.2. Optimal consumption and wealth

Since there are only two states, no friction, no restrictions on agents’ portfolios and agents have equivalent beliefs, the stock and the bond are enough to ensure complete markets. Moreover, as explained in Blume and Easley (2002), the equilibrium is Pareto efficient. Due to the Inada conditions, starting in state $i$, the optimality condition states that the marginal utility of each investor must be equal across states so

$$\lambda_a p_{ija}(s)c_{ija}^{-\gamma}(s) = \lambda_b p_{ijb}(s)c_{ijb}^{-\gamma}(s).$$

The equilibrium condition for the good market is

$$c_{ija}(s) + c_{ijb}(s) = y_j.$$

Hence

$$c_{ija}(s) = \frac{(\lambda_a p_{ija}(s))^{\frac{1}{\gamma}}y_j}{(\lambda_a p_{ija}(s))^{\frac{1}{\gamma}} + (\lambda_b p_{ijb}(s))^{\frac{1}{\gamma}}}.$$

(2.2)

where $\lambda_a$ and $\lambda_b$ are non-negative constants representing the inverse of the Lagrange multipliers for each individual problem. It is always possible to normalize them so that their sum is equal to one

$$\lambda_a + \lambda_b = 1.$$

They are sufficient statistics that capture the relative importance of the two populations on the determination of the equilibrium optimal allocations and prices. Inspection of relationship (2.2) reveals that agents’ decisions are time inconsistent. At time 0, agent $a$ optimally chooses her consumption plan for time $s$ according to relationship (2.2). However, at time $s$, in state $j$, agent $a$ would like to depart from her plan and consume

$$c_{ja} = \frac{\frac{1}{\gamma} \lambda_a y_j}{\lambda_a^{\frac{1}{\gamma}} + \lambda_b^{\frac{1}{\gamma}}}.$$

For convenient reason, define the weights

$$w_a = \frac{\lambda_a^{\frac{1}{\gamma}}}{\lambda_a^{\frac{1}{\gamma}} + \lambda_b^{\frac{1}{\gamma}}} \text{ and } w_b = 1 - w_a.$$

At time $t$ in state $i$, the wealth of agent $a$ is given by the cumulated stream of future consumption plans priced at the state price density $\frac{u'(c_{k}(s))}{u'(c_{k}(t))} \beta^{(s-t)}$, i.e.

$$W_{ia}(t) = \frac{1}{u'(c_{ia}(t))} E_{i,a,t}^{i} \left[ \sum_{s=t+1}^{\infty} \beta^{(s-t)}u'(c_{a}(s))c_{a}(s) \right],$$

10
\[ W_{ia}(t) = \sum_{s=t+1}^{\infty} \beta^{s-t} \left( w_a \frac{1}{y_i} \left( w_a p_{iia}(s) + w_b p_{iib}(s) \right)^{y_i} \right) \]  

\[ + w_a p_{ija}(s) \left( w_a p_{ija}(s) + w_b p_{ijb}(s) \right)^{y_{ij}} \left( \frac{y_i}{y_j} \right)^{y_j} \]  

(2.3)

### 2.3.3. Equilibrium asset price and interest rate

**Asset price**  
The equilibrium asset price \( S_i \) in state \( i \) is given by

\[ S_i = \frac{1}{u'(c_{ik}(0))} E_k \left[ \sum_{s=1}^{\infty} \beta^s u'(c_{ik}(s)) y(s) \right], \]

where \( k \in \{a, b\} \) so we have

\[ S_i = \sum_{s=1}^{\infty} \beta^s \left( \left( w_a p_{iia}(s) + w_b p_{iib}(s) \right)^{y_i} + \left( w_a p_{ija}(s) + w_b p_{ijb}(s) \right)^{y_{ij}} \left( \frac{y_i}{y_j} \right)^{y_j} \right). \]  

(2.4)

Relationship (2.4) shows that price \( S_i \) is a continuous function of the weight \( \lambda_a \). If \( \lambda_a > \lambda_b \), the equilibrium price \( S_i \) is close to \( S_{ia} \). For this reason, Varian (1989) calls the expression \( \lambda_a p_{ija} \) the weighted probability.

The equilibrium exists and is unique and the weight \( \lambda_a \) is a constant determined by agent \( a \) budget constraint (see for instance Wang (1996)), i.e. \( \lambda_a \) is the unique solution (see appendix 1) of the equation

\[ z_{ia}^- S_i = W_{ia}(0), \]  

(2.5a)

where \( S_i \) and \( W_{ia}(0) \) are given respectively by relationships (2.4) and (2.3). Notice that relationships (2.3) and (2.4) are smooth in the variables \( (z_{ia}^-; \gamma) \) and therefore, \( \lambda_a \) is also smooth in \( z_{ia}^- \) and \( \gamma \).

An important case is when the two classes are “equally” represented \( \lambda_a = \lambda_b = \frac{1}{2} \), which occurs exactly when the initial endowment \( z_{ia}^- \) (allocated in state \( i \)) is such that

\[ z_{ia}^- = \sum_{s=1}^{\infty} \beta^s \left( \left( p_{iia}(s) + p_{iib}(s) \right)^{y_i} + \left( p_{ija}(s) + p_{ijb}(s) \right)^{y_{ij}} \left( \frac{y_i}{y_j} \right)^{y_j} \right), \]  

\[ \sum_{s=1}^{\infty} \beta^s \left( \left( p_{iia}(s) + p_{iib}(s) \right)^{y_i} \right)^{y_i} + \left( \frac{p_{ija}(s) + p_{ijb}(s)}{y_j} \right)^{y_j} \]  

(2.5a)
Equilibrium interest rate  The equilibrium interest rate $r_i$ in state $i \in \{1, 2\}$ satisfies

$$\frac{1}{1 + r_i} u'(c_{ik}(t)) = \beta E_{k,t}^i \left[ u'(c_{ik}(t + 1)) \right],$$

for $k = a, b$. As shown in appendix 1, we have

$$\frac{1}{1 + r_i} = \beta \left[ \left( w_a (1 - \phi_{ia}) \right)^\gamma + w_b (1 - \phi_{ib}) \right] \gamma + \left( w_a \phi_{ia} + w_b \phi_{ib} \right) \gamma \left( \frac{y_i}{y_j} \right) \gamma.$$ 

2.3.4. Portfolio allocations

Recall that the dynamics of agent $a'$s wealth are given by

$$W_k(t + 1) - W_k(t) = x_k(t)r(t) - c_k(t) + z_k(t) (y(t) + S(t + 1) - S(t)).$$

Hence, being in state $i$, the optimal number of shares held by agent $a$ is

$$z_{ia} = \frac{r_i W_{ia} - c_{ia}}{r_i S_i - y_i}.$$ 

2.3.5. Myopic Investors

In this section, we assume that agents have logarithmic preferences

$$u(c) = \log c.$$ 

Given what precedes, we find that the equilibrium price is independent of beliefs since

$$S_i = \frac{y_i}{\theta}.$$ 

From relationship (2.3), agent $k$ wealth is given by

$$W_{ia} = w_{ia} \frac{y_i}{\theta}.$$ 

Since $W_{ia} = z_{ia}^- S_i$, we conclude that $w_a = \lambda_a = z_{ia}^-$. No trade is the equilibrium: agents consume what they can afford given their initial endowment, $c_{ia} = z_{ia}^- y_i$. Finally, the equilibrium interest rate $r_i$ is such that

$$\frac{1}{1 + r_i} = \frac{z_{ia}^-}{1 + r_{ia}} + \frac{z_{ib}^-}{1 + r_{ib}}.$$ 

In this case, autarky is a time consistent stationary equilibrium. It can be shown that only under logarithmic preference such an equilibrium exists.
3. Discussion

The aim of this section is to investigate under which conditions on preferences and beliefs a speculation phenomenon as related in Harrison and Kreps (1978) may arise. We first start with the following lemma.

Lemma 1. For $\lambda$ in $[0, 1]$, $(x, y)$ in $\mathbb{R}_+^2$ and $\gamma \geq 1$ ($\gamma \leq 1$), we have

$$
\frac{(\lambda x)^{\frac{1}{\gamma}} + ((1-\lambda)y)^{\frac{1}{\gamma}}}{(\lambda x + (1-\lambda)y)^{\frac{1}{\gamma}}} \leq (\geq) \frac{\lambda^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} + (1-\lambda)^{\frac{1}{\gamma}}} x + \frac{(1-\lambda)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} + (1-\lambda)^{\frac{1}{\gamma}}} y,
$$

with strict inequality for $\gamma \neq 1$.

Proof. See appendix 2. ■

Given relationship (2.4), it follows that

$$S_i < (> w_a S_{ia} + w_b S_{ib} \text{ if } \gamma > 1(\gamma < 1).$$

Inspection of the previous relationship leads to the following proposition.

Proposition 1. When individuals are more risk averse than the myopic investor ($\gamma > 1$), no speculation phenomenon takes place.

Conversely, when investors are not too risk averse ($\gamma < 1$), the asset equilibrium price may lie above the maximum of agents’ fundamental valuations. Being less risk averse than the myopic investor is a necessary but not sufficient condition for speculation. For instance, if $S_{ia}$ and $S_{ib}$ are close enough, then speculation occurs, at least in one state. One can verify that

$$S_{ia} = S_{ib} \iff \frac{\phi_{ia}}{\theta + \phi_{ia} + \phi_{ja}} = \frac{\phi_{ib}}{\theta + \phi_{ib} + \phi_{jb}},$$

independently of $\gamma$, $y_1$ and $y_2$. Let us assume that $S_{ia} = S_{ib}$, so that

$$\theta = \frac{\phi_{ib} \phi_{2a} - \phi_{ia} \phi_{2b}}{\phi_{ia} - \phi_{ib}}.$$ 

Given what precedes, if $\gamma < 1$

$$S_1 > \max \{S_{1a}, S_{1b}\},$$

i.e. speculation occurs in state 1; however, for arbitrary risk aversion coefficient $\gamma$, in general, we cannot guarantee that it also occurs in state 2.

We now investigate how equilibrium prices and allocations behave when agents become asymptotically risk neutral. We begin by establishing another preliminary result.

\footnote{We cannot have $S_{ia} = S_{ib}$, for $i = 1, 2$, unless $\phi_{ia} = \phi_{ib}$ for $i = 1, 2$, i.e. agents have common beliefs.}

\footnote{Notice that since we require $\theta$ to be positive, without loss of generality assuming that $\phi_{1a} > \phi_{1b}$, we must have $\frac{\phi_{1a}}{\phi_{2b}} > \frac{\phi_{1a}}{\phi_{2a}}$, which means that beliefs must cross.}
Lemma 2. Let $f$ and $g$ be two continuous real value functions. Then
\[
\lim_{\gamma \to 0} \left( f(\gamma)^{\frac{1}{\gamma}} + g(\gamma)^{\frac{1}{\gamma}} \right)^\gamma = \max \{ f(0), g(0) \}.
\]

Proof. See appendix 2. ■

Corollary 1.
\[
\lim_{\gamma \to 0} \frac{1}{f(0)} f(\gamma)^{\frac{1}{\gamma}} \left( f(\gamma)^{\frac{1}{\gamma}} + g(\gamma)^{\frac{1}{\gamma}} \right)^{-1} = \begin{cases} 
1_{\{f(0) > g(0)\}} f(0), & f(0) \neq g(0) \\
\frac{1}{2}, & f(0) = g(0),
\end{cases}
\]
where $1_A$ is the indicator function\(^6\).

We can now investigate the equilibrium allocations and prices when agents are asymptotically risk neutral.

3.1. Asymptotic risk neutral investors

3.1.1. Stock market and optimal allocations

In the limit, when investors become risk neutral (i.e. $\gamma \to 0$), using Lebesgue dominated theorem and the fact that the set $\{ t \geq 0, p_{ija}(t) = p_{ijb}(t) \}$ has measure zero, we obtain that
\[
S_i = \sum_{s=1}^{\infty} \beta^s \left( \max \{ \lambda_{a} p_{iia}(s), \lambda_{b} p_{iib}(s) \} y_i + \max \{ \lambda_{a} p_{ija}(s), \lambda_{b} p_{ijb}(s) \} y_j \right) \max \{ \lambda_{a}, \lambda_{b} \}.
\]

The corresponding wealth is
\[
W_{ia} = \sum_{s=1}^{\infty} \beta^s \left( \Delta(\lambda_{a} p_{iia}(s), \lambda_{b} p_{iib}(s)) y_i + \Delta(\lambda_{a} p_{ija}(s), \lambda_{b} p_{ijb}(s)) y_j \right) \max \{ \lambda_{a}, \lambda_{b} \},
\]
where the function $\Delta$ is defined by
\[
\Delta(x, y) = \begin{cases} 
x, & \text{if } x > y \\
\frac{x}{y}, & \text{if } x = y \\
0, & \text{if } x < y.
\end{cases}
\]

The optimal planned consumption allocations are
\[
\lim_{\gamma \to 0} c_{ija}(s) = 1_{\{ \lambda_{a} p_{ija}(s) > \lambda_{b} p_{ijb}(s) \}} y_i
\]
\[
\lim_{\gamma \to 0} c_{ijb}(s) = 1_{\{ \lambda_{a} p_{ija}(s) < \lambda_{b} p_{ijb}(s) \}} y_j.
\]

Optimal planned consumptions are half of the fruit when on the set $\{ \lambda_{a} p_{ija}(s) = \lambda_{b} p_{ijb}(s) \}$.

\(^6\)The indicator function $1_A$ is defined as $1_A(a) = 1$ if $a \in A$ and 0 otherwise.
3.1.2. Interest rate

Given what precedes, the equilibrium interest rate is given by

\[
\frac{1}{1 + r_i} = \beta \left[ \max \{\lambda_a \phi_{ia}, \lambda_b \phi_{ib}\} + \max \{\lambda_a(1 - \phi_{ia}), \lambda_b(1 - \phi_{ib})\} \right].
\]

If \( \lambda_a = \frac{1}{2} \)

\[
\frac{1}{1 + r_i} = \beta [1 + |\phi_{ia} - \phi_{ib}|].
\]

As shown in appendix, we always have \( r_i \leq \theta \) with possibly strict inequality. Hence, the equilibrium price of the bond always lies below its corresponding value in an economy populated by homogenous agents. This reflects the fact that risk neutral investors with heterogeneous beliefs have a stronger incentive to hold the stock than holding the bond with respect to an economy populated by homogenous individuals.

3.1.3. Speculation

If the initial wealth distribution is adjusted such that \( \lambda_a = \lambda_b = \frac{1}{2} \), then

\[
S_i = \sum_{s=1}^{\infty} \beta^s \left( \max \{p_{ia}(s), p_{ib}(s)\}y_i + \max \{p_{ija}(s), p_{ijb}(s)\}y_j \right).
\]

(3.3)

Inspection of relationship (3.3) reveals that if investors have different beliefs, then we have

\[ S_i > \max \{S_{ia}, S_{ib}\}. \]

By continuity in \( \gamma \), for \( \gamma \) positive close enough to 0, the speculation phenomenon still holds for risk averse investors. This means that when investors are not too risk adverse and have heterogeneous beliefs, speculation arises. To fix idea, we now examine in details an example.

3.1.4. Example

Let us assume that

\[ \phi_{1a} + \phi_{2a} = \phi_{1b} + \phi_{2b}, \]

and we initially start in state 1 with a wealth distribution such that \( \lambda_a = \lambda_b = \frac{1}{2} \). We also assume that \( \phi_{1b} > \phi_{1a} \), i.e. \( \phi_{2a} > \phi_{2b} \); agent \( b \) (a) thinks that the average duration of state 1 (2) is lower than what agent \( a \) (b) thinks. It follows that for all \( t \geq 0 \)

\[ p_{11a}(t) > p_{11b}(t) \text{ and } p_{22b}(t) > p_{22a}(t). \]

Equilibrium prices of the tree in states 1 and 2 are given by

\[
S_1 = \frac{(\theta(1 - \phi_{1a}) + \phi_{2a})y_1 + \phi_{1b}(1 + \theta)y_2}{\theta(\theta + \phi_{1a} + \phi_{2a})},
\]

\[
S_2 = \frac{(\theta + \phi_{2a}y_1 + (\theta(1 - \phi_{2b}) + \phi_{1b})y_2}{\theta(\theta + \phi_{1a} + \phi_{2a})},
\]

15
we have

Investors’ positions when no switch occurs are reported in Table I and Table II.

risky asset when a switch of state occurs. We consider two sets of values for agents’ beliefs.

values

Numerical simulations

and the interest rate is such that

\[
\frac{1}{1 + r_i} = \frac{1}{1 + r_j} = \beta [1 + |\phi_{ia} - \phi_{ib}|].
\]

The corresponding optimal wealths are

\[
W_{1a} = \frac{(\theta(1-\phi_{ib}) + \phi_{ib})y_1}{\theta(\theta + \phi_{ia} + \phi_{ib})}, \quad W_{1b} = \frac{(1+\theta)\phi_{ib}y_2}{\theta(\theta + \phi_{ia} + \phi_{ib})},
\]

\[
W_{2a} = \frac{(\theta+\phi_{ia}+\phi_{ib})y_1}{(1+\theta)(\theta+\phi_{ia}+\phi_{ib})}, \quad W_{2b} = \frac{(\theta(1-\phi_{ib}) + \phi_{ib})y_2}{\theta(\theta + \phi_{ia} + \phi_{ib})},
\]

and the optimal consumptions and portfolio allocations are

\[
\begin{align*}
 z_{12a} &= -\frac{y_1}{y_2-y_1}, & z_{12b} &= \frac{y_2}{y_2-y_1}, & c_{1a} = \frac{y_1}{y_2}, & c_{1b} = \frac{y_1}{y_2}, \\
 z_{21a} &= -\frac{y_1}{y_2-y_1}, & z_{21b} &= \frac{y_2}{y_2-y_1}, & c_{2a} = \frac{y_2}{y_2}, & c_{2b} = \frac{y_2}{y_2}.
\end{align*}
\]

Positions \(z_{ii}\) and \(z_{ib}\) are discussed below using numerical simulations. When \(\theta\) goes to zero we have

\[
\begin{align*}
 z_{ii} &= \frac{\phi_{ib}y_2}{\phi_{ia}y_1 + \phi_{ib}y_2}, & z_{ib} &= \frac{\phi_{ib}y_2}{\phi_{ia}y_1 + \phi_{ib}y_2},
\end{align*}
\]

for \(i = 1, 2\). This implies that

\[
\begin{align*}
 x_{11a} &= \frac{y_1(\phi_{ib}y_2 - \phi_{ia}y_1)}{(\phi_{ia}y_1 + \phi_{ib}y_2)(\phi_{ia} - \phi_{ib})}, & x_{11b} &= -\frac{y_1(\phi_{ib}y_2 - \phi_{ia}y_1)}{(\phi_{ia}y_1 + \phi_{ib}y_2)(\phi_{ia} - \phi_{ib})}, \\
 x_{22a} &= \frac{y_2(\phi_{ib}y_2 - \phi_{ia}y_1)}{(\phi_{ia}y_1 + \phi_{ib}y_2)(\phi_{ia} - \phi_{ib})}, & x_{22b} &= -\frac{y_2(\phi_{ib}y_2 - \phi_{ia}y_1)}{(\phi_{ia}y_1 + \phi_{ib}y_2)(\phi_{ia} - \phi_{ib})}.
\end{align*}
\]

**Numerical simulations.** The time discount rate \(\theta\) is kept as a free parameter. We assign values \(y_1 = 1\) and \(y_2 = 2\). It follows that \(z_{12a} = z_{21a} = -\frac{1}{2}\), so agent \(a\) always shorts the risky asset when a switch of state occurs. We consider two sets of values for agents’ beliefs. Investors’ positions when no switch occurs are reported in Table I and Table II.

**Table I: Impact of the time discount rate on agents’ positions**

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>0</th>
<th>0.01</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_{11a})</td>
<td>0.333</td>
<td>0.331</td>
<td>0.342</td>
<td>0.564</td>
<td>0.902</td>
<td>3.740</td>
</tr>
<tr>
<td>(x_{11a})</td>
<td>1.666</td>
<td>1.535</td>
<td>0.789</td>
<td>-0.106</td>
<td>-0.366</td>
<td>-0.635</td>
</tr>
<tr>
<td>(z_{22a})</td>
<td>0.333</td>
<td>0.311</td>
<td>0.074</td>
<td>-1.409</td>
<td>-3.625</td>
<td>-23.179</td>
</tr>
<tr>
<td>(x_{22a})</td>
<td>3.333</td>
<td>3.443</td>
<td>4.286</td>
<td>6.363</td>
<td>7.5</td>
<td>9.286</td>
</tr>
</tbody>
</table>

| \(\phi_{ia} = 0.2, \phi_{2a} = 0.4, \phi_{1b} = 0.4, \phi_{2b} = 0.2.\) |

**Table II: Impact of the time discount rate on agents’ positions**

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>0</th>
<th>0.01</th>
<th>0.085</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_{11a})</td>
<td>0.680</td>
<td>0.737</td>
<td>1.039</td>
<td>1.085</td>
<td>1.978</td>
<td>2.926</td>
<td>10.237</td>
</tr>
<tr>
<td>(x_{11a})</td>
<td>-7.20</td>
<td>-6.765</td>
<td>-4.897</td>
<td>-4.681</td>
<td>-2.815</td>
<td>-2.367</td>
<td>-1.938</td>
</tr>
<tr>
<td>(z_{22a})</td>
<td>0.680</td>
<td>0.810</td>
<td>-0.045</td>
<td>0.097</td>
<td>0.953</td>
<td>1.688</td>
<td>7.417</td>
</tr>
<tr>
<td>(x_{22a})</td>
<td>-14.40</td>
<td>-17.732</td>
<td>9.912</td>
<td>6.452</td>
<td>-1.726</td>
<td>-2.318</td>
<td>-2.753</td>
</tr>
</tbody>
</table>
$\phi_{1a} = 0.15$, $\phi_{2a} = 0.85$, $\phi_{1b} = 0.2$, $\phi_{2b} = 0.8$.

We notice that the time discount factor $\theta$ plays a significant role on the positions taken by both investors. For small values of $\theta$, both investors have long positions. In table I, as $\theta$ goes up, agent $a$ increases her long position in state 1 by borrowing from agent $b$. Alternatively, in state 2, she increases her short position, lending to agent $b$. Conversely, in table II, agent $a$ increases her long position both in states 1 and 2 by borrowing from agent $b$.

Finally, the initial endowment $z_{1a}^-$ (allocated in state 1) of agent $a$ is such that $z_{1a}^- S_1 = W_{1a}$. We find that

$$z_{1a}^- = \frac{(\theta + \phi_{2a})y_1}{(\theta + \phi_{2a})y_1 + \phi_{1b}y_2}.$$ 

It is easy to see that $z_{1a}^-$ is increasing with $\theta$ from $\frac{\phi_{2a}y_1}{\phi_{2a}y_1 + \phi_{1b}y_2}$ up to 1. To conclude the discussion, we now investigate the case of asymptotically infinitely risk averse investors.

### 3.2. No speculation

We have already established that when $\gamma \geq 1$, speculation does not take place. We now investigate the possibility for the asset equilibrium price to lie below the minimum of the two fundamental valuations, at least in one state. To do so, we look at the extreme case when investors become asymptotically infinitely risk averse. The results are summarized in the next proposition.

**Proposition 2.** At the equilibrium, no trade takes place between the two classes of investors. The class of investors endowed with $z_k < \frac{1}{2}$ shares in state $i$ asymptotically vanishes and its weigh $\lambda_k$ is such that

$$\lambda_k(\gamma) \sim \frac{z_k^-}{1 - z_k^-}.$$ 

The equilibrium asset prices are

$$\lim_{\gamma \to \infty} S_1 = \left( \sum_{s=1}^{\infty} \beta_s p_{11a}(s) p_{11b}(s) \right) y_1,$$

$$\lim_{\gamma \to \infty} S_2 = \infty,$$

$$\lim_{\gamma \to \infty} \frac{S_1}{\sum_{s=1}^{\infty} \beta_s p_{iia}(s) p_{iib}(s)} = \frac{\sum_{s=1}^{\infty} \beta_s p_{iia}(s) p_{iib}(s)}{\sum_{s=1}^{\infty} \beta_s p_{iia}(s)} z_{1a} \leq 1.$$
with strict inequality whenever \( p_{iia} \neq p_{iib} \). Moreover,
\[
\lim_{\gamma \to \infty} \frac{1}{1 + r_1} = \beta(1 - \phi_{1a})^z_{1a}(1 - \phi_{1b})^z_{1b} \\
\lim_{\gamma \to \infty} r_2 = -1 \\
\lim_{\gamma \to \infty} \frac{1 + r_{2a}}{1 + r_2} = \left(\frac{\phi_{2b}}{\phi_{2a}}\right)^z_{1b} \quad \text{and} \quad \lim_{\gamma \to \infty} \frac{1 + r_{2b}}{1 + r_2} = \left(\frac{\phi_{2a}}{\phi_{2b}}\right)^z_{1a},
\]
and
\[
\lim_{\gamma \to \infty} c_{ijk} = \bar{z}_k y_j \\
\lim_{\gamma \to \infty} z_{ik} = \lim_{\gamma \to \infty} z_{jk} = \bar{z}_k.
\]

**Proof.** See appendix 3. □

Infinitely risk averse agents do not trade among each other. Now, consider the case where
\[
\frac{\phi_{1a}}{\theta + \phi_{1a} + \phi_{2a}} = \frac{\phi_{1b}}{\theta + \phi_{1b} + \phi_{2b}},
\]
or equivalently
\[
\theta = \frac{\phi_{1b} \phi_{2a} - \phi_{1a} \phi_{2b}}{\phi_{1a} - \phi_{1b}}.
\]
In particular, if \( \phi_{1a} > \phi_{1b} \), we must have \( \frac{\phi_{2a}}{\phi_{2b}} > \frac{\phi_{1a}}{\phi_{1b}} \) in order to have \( \theta > 0 \). This rules out the possibility that \( p_{11b}(s) > p_{11a}(s) \) for all \( s \in \mathbb{R}_+ \). It follows that
\[
S_{1a} = S_{1b}.
\]
It follows that
\[
S_1 < \min \{ S_{1a}, S_{1b} \}.
\]
When agents become infinitely risk averse, the economy converges to autarchy. Due to high risk aversion, the market is a bear market: each investor try to get rid of shares and at the equilibrium the two investors do not have any interaction with each other (they do not lend or borrow from each other). Both investors consume and own according to their initial endowment. Each investor is asking for strictly less than one share (instead of exactly one share), which drives the equilibrium asset price down with respect to an economy populated with only one type of agent. This result can be related to Varian (1985) who shows that if risk aversion does not decline too quickly, then an increase in diversity of opinion will lead to a decrease in the asset price. Abel (1989) also show that an increase in belief heterogeneity can lower the asset price. By continuity, the result still holds for large values of \( \gamma \).

**4. Extensions to the basic model**

We present three extensions of the basic model for which we examine how equilibrium allocations and prices behave when at least one agent becomes risk neutral in the limit.
4.1. Many agents

In this paragraph, we assume that the economy is populated by \( N \) agents with heterogeneous beliefs but identical CRRA preferences

\[
u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}.
\]

For \( k = 1, 2, ..., N \), let \( \phi_{ik} \) denote the probability assigned by agent \( k \) of switching from state \( i \) to state \( j \) and \( p_{ijk}(s) \) the corresponding conditional probability of being in state \( j \) at time \( s \), being in state \( i \) at time 0. Starting in state \( i \), the optimality conditions are

\[
\lambda_k p_{ijk}(t)c_{ijk}^{-\gamma}(t) = \lambda_h p_{ijh}(t)c_{ijh}^{-\gamma}(t),
\]

for all \( \{k, h\} \in \{1, 2, ..., N\}^2 \), where \( \lambda_k \) and \( \lambda_h \) are non-negative constant with the normalization

\[
\sum_{k=1}^{N} \lambda_k = 1.
\]

Combined with the equilibrium condition on the goods market

\[
\sum_{k=1}^{N} c_{ijk} = y_j,
\]

for \( k = 1, 2, ..., N \), we find that

\[
c_{ijk}(t) = \frac{(\lambda_k p_{ijk}(t))^{\frac{1}{\gamma}} y_j}{\sum_{k=1}^{N} (\lambda_k p_{ijk}(t))^{\frac{1}{\gamma}}}
\]

4.1.1. Asymptotically risk neutral agents

Lemma 2. can easily be generalized to the case of \( N \) variables so

\[
\lim_{\gamma \to 0} p_{ijk}(t)c_{ijk}^{-\gamma}(t) = \max_{k \in \{1,2,\ldots,N\}} \{\lambda_k p_{ijk}(t)\},
\]

assuming that agents have distinct beliefs. Hence the equilibrium price of the tree in state \( i \) is given by

\[
S_i = \frac{\sum_{s=1}^{\infty} \beta^s \left( \max_{k \in \{1,2,\ldots,N\}} \{\lambda_k p_{ik}(s)\} y_i + \max_{k \in \{1,2,\ldots,N\}} \{\lambda_k p_{ijk}(s)\} y_j \right)}{\max_{k \in \{1,2,\ldots,N\}} \{\lambda_k\}}.
\]

Assume that the initial wealth endowments are such that every investor has equal weight \( \lambda_k = \frac{1}{N} \) for all \( k = 1, 2, ..., N \). Inspection of (4.2) reveals that more participants in the
market with distinct beliefs enhances the speculation phenomenon with respect to the two person case. Moreover, we have

$$\lim_{\gamma \to 0} c_{ijk}(t) = 1_{A_{ijk},y_j},$$

where $A_{ijk} = \{\lambda_k p_{ijk}(t) = \max_{h \in \{1,2,\ldots,N\}} \lambda_h p_{ijh}(t)\}$. This is a clear extension of results obtained when $N = 2$. In particular, no speculation phenomenon occurs when $\gamma \geq 1$. As before, the weights $\lambda_k$ are determined by $N - 1$ conditions given by agents' initial wealths that must satisfy

$$W_{ik}(0) = z_{ik}^{-1}S_i, \text{ for } k = 1, 2, \ldots, N - 1,$$

where

$$W_{ik}(0) = \frac{1}{\max_{h \in \{1,2,\ldots,N\}} \{\lambda_h\}} \sum_{s=1}^{\infty} \beta^s \left(1_{A_{iik}} \lambda_k p_{iik}(s) y_i + 1_{A_{ijk}} \lambda_k p_{ijk}(s) y_j\right),$$

and the normalization relationship (4.1). As before, the optimal number of shares held by agent $k$ in state $i$ is

$$z_{ik} = \frac{r_i W_{ik} - c_{ik}}{r_i S_i - y_i}.$$

The interest rate is given by

$$\frac{1}{1 + r_i} = \beta \left[ \left( \sum_{k=1}^{N} w_k (1 - \phi_{ik})^{\frac{1}{\gamma}} \right)^{\gamma} + \left( \sum_{k=1}^{N} w_k \phi_{ik}^{\frac{1}{\gamma}} \right)^{\gamma} \left( \frac{y_i}{y_j} \right)^{\gamma} \right],$$

where weight $w_k$ is given by

$$w_k = \frac{\lambda_k^{\frac{1}{\gamma}}}{\sum_{k=1}^{N} \lambda_k^{\frac{1}{\gamma}}}.$$

Finally, since lemma 1. uses a convexity argument, it can be easily extended to the case of $N$ agents and therefore the necessary but not sufficient condition to get speculation, i.e., $\gamma < 1$ still holds.

### 4.2. Two heterogeneous CRRA preference agents

In this paragraph, we assume that agent $a$ has preferences represented by the following CRRA utility function

$$u_a(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha},$$

with $\alpha > 0$. As before, agent $b$'s utility function is

$$u_b(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}.$$
Starting in state $i$, the optimal and equilibrium conditions respectively are
\[
\lambda_a p_{ija}(t) c_{ija}(t) = \lambda_b p_{ijb}(t) c_{ijb}(t)
\]
\[
c_{ija}(t) + c_{ijb}(t) = y_j.
\]
It follows that
\[
c_{ija}(t) (\lambda_b p_{ijb})^{\frac{1}{\gamma}} + c_{ijb} (\lambda_a p_{ija})^{\frac{1}{\gamma}} = (\lambda_a p_{ija})^{\frac{1}{\gamma}} y_j.
\]
Unfortunately, it is not possible to get a closed form expression of the optimal allocations for all values of $\gamma$. However, we are mainly interesting in getting some insights about the optimal allocations when the agent $b$ is nearly risk neutral. It turns out at the limit when $\gamma$ goes to 0 we can fully characterize the equilibrium, and therefore, by continuity of the parameter $\gamma$ have some information about the equilibrium structure when the second investor is nearly risk neutral. As proved in appendix 4., we have
\[
\lim_{\gamma \to 0} p_{ija} c_{ija}^{-\alpha} = a.s. \max \{ p_{ija} y_j^{-\alpha}, \frac{\lambda_b p_{ijb}}{\lambda_a} \}.
\]
Hence the equilibrium price of the tree in state $i$ is given by
\[
S_i = \max \{ \frac{\lambda_a y_i^{-\alpha}}{1 + y_i^{-\alpha}} \}
\]
Note that when $\lambda_a$ is equal to 0 or 1, the above expression for $S_i$ gives us the equilibrium prices of the tree for an populated by only one class of agent. In addition, it is easy to check that when investors have common beliefs, the equilibrium asset price $S_i$ lies within the range of prices determined when agents are alone in the economy. Hence, differences in beliefs are indispensable for speculation. The critical value for not having one class of agents under-represented is not longer $\lambda_a = \frac{1}{2}$ but instead
\[
\lambda_a = \frac{y_i^{-\alpha}}{1 + y_i^{-\alpha}}.
\]
Finally, investors’ optimal planned consumption are
\[
\lim_{\gamma \to 0} c_{ija}(t) = \min \{ \frac{\lambda_a p_{ija}(t)}{\lambda_b p_{ijb}(t)} y_j \}
\]
\[
\lim_{\gamma \to 0} c_{ijb}(t) = \max \{ y_j - \frac{\lambda_a p_{ija}(t)}{\lambda_b p_{ijb}(t)} y_i \}
\]
Agent $a$’s wealth is given by
\[
W_{ia} = \frac{1}{\max \{ \frac{\lambda_a y_i^{-\alpha}}{1 + y_i^{-\alpha}}, \frac{\lambda_b}{\lambda_b y_i^{-\alpha}} \}} \sum_{s=1}^{\infty} \beta^s \left( \max \{ \frac{\lambda_a p_{ija}(s) y_i^{-\alpha}}{\lambda_b p_{ijb}(s)} \} \min \{ \frac{\lambda_a p_{ija}(s)}{\lambda_b p_{ijb}(s)} y_i \} \right.
\]
\[
+ \max \{ \lambda_a p_{ija}(s) y_j^{-\alpha}, (\lambda_b p_{ijb}(s)) \} \min \{ \frac{\lambda_a p_{ija}(s)}{\lambda_b p_{ijb}(s)} y_j \}
\]
As before, the optimal portfolio holdings are

\[ z_{ia} = \frac{r_iW_{ia} - c_{ia}}{r_iS_i - y_i}. \]

5. Learning

As developed in Morris (1996), it seems plausible to assume that investors might dynamically revise their assessment as new information arrives. In what precedes, we have taken beliefs as given. We now relax this assumption and allow individuals to revise their beliefs about the transition probabilities \( \mu_1 \) and \( \mu_2 \) upon the arrival of state changes. To get some insights, we first consider the case of Bernoulli beliefs. We then sketch how to deal with general beliefs.

5.1. Information Structure

Agents do not know the true values of the transition probabilities \( \mu_1 \) and \( \mu_2 \). However, they do know that they are constant and they hesitate over two values

\[ \mu_i \in \{ \mu_{il}, \mu_{ih} \}. \]

Let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by the observations of the dividend process \( \{y(s), 0 \leq s \leq t\} \) and augmented. At time \( t \), the investor’s common information set is \( \mathcal{F}_t \).\(^7\) At time \( t \), let \( q_{ik}(t) \) be the probability or the investor \( k \)’s belief that \( i \) equals \( \mu_{ih} \), i.e., \( q_{ik}(t) = \Pr(\mu_i = \mu_{ih} \mid \mathcal{F}_t) \). The evolution across time of the posterior probability \( q_{ik} \) is given by the following lemma.

Lemma 3. Being in state \( i \in \{1, 2\} \), the law of motion of the posterior belief \( q_{ik} \) is given by

\[
q_{ik}(t + 1) = \begin{cases} 
q_{ik}(t)(1-\mu_{ih}) 
& \text{if stay in state } i \\
q_{ik}(t)(1-\mu_{ih})/(1-q_{ik}(t)(1-\mu_{ih})) 
& \text{if switch into state } j 
\end{cases}
\]

Proof. See appendix 5. \( \Box \)

Since, \( q_i \) is the probability assigned to the highest value for \( \mu_i \), if the state of the world remains the same, the investor revised downward her beliefs; however, should a switch occurs, she increments upward her beliefs. Obviously, being in state \( i \) provides no information about \( \mu_j \), so as long as state \( i \) prevails, beliefs \( q_{jk} \) remain unchanged. It is worth noticing (and easy to check) that no beliefs overtaking can occur, i.e. if at some date

\(^7\)The filtration \( F = \{ \mathcal{F}_t, t \in \mathbb{R}_+ \} \) is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented).
Let $q_{ia}(t) < q_{ib}(t)$, then for all dates $s \geq t$, we have $q_{ia}(s) < q_{ib}(s)$. Finally, note that $q_{ik}$ is a martingale under the investor’s belief $P_k$ since

$$E^i_{k,t}(q_{ik}(t + 1)) = (1 - q_{ik}(t))\mu_{ih} + (1 - q_{ik}(t))\mu_{il} \times \frac{q_{ik}(t)(1 - \mu_{ih})}{q_{ik}(t)(1 - \mu_{ih}) + (1 - q_{ik}(t))(1 - \mu_{il})}$$

$$+ (q_{ik}(t))\mu_{ih} + (1 - q_{ik}(t))\mu_{il} \times \frac{q_{ik}(t)\mu_{ih} + (1 - q_{ik}(t))\mu_{il}}{q_{ik}(t)\mu_{ih} + (1 - q_{ik}(t))\mu_{il}} = q_{ik}(t).$$

It is also possible to check that, under the probability measure $P_l (P_h)$, $p_{ik}$ is a super-martingale (submartingale). In order to get some insight about the equilibrium price, we first examine the case of an economy populated by homogenous individuals.

**5.2. Homogenous investors**

Let $E^m_{i,t}$ denote the conditional expectation at time $t$, being in state $i$ and knowing that the intensity of switching from state $i$ ($j$) into state $j$ ($i$) is $\mu_{im}$ ($\mu_{jn}$). Then at time 0, being in state $i$ with initial beliefs $(q_i, q_j)$, the equilibrium price of the tree is given by

$$S_i = \frac{1}{u'_{i}(y_i)} E^a_{i,t} \left[ \sum_{s=t+1}^{\infty} \beta^{(s-t)} u'(y(s))y(s) \right]$$

$$= \frac{1}{u'_{i}(y_i)} \left( q_i q_j E_{i,t}^{h,h} \left[ \sum_{s=t+1}^{\infty} \beta^{(s-t)} u'(y(s))y(s) \right] + q_i (1 - q_j) E_{i,t}^{h,d} \left[ \sum_{s=t+1}^{\infty} \beta^{(s-t)} u'(y(s))y(s) \right] \right)$$

$$+ (1 - q_i) q_j E_{i,t}^{d,h} \left[ \sum_{s=t+1}^{\infty} \beta^{(s-t)} u'(y(s))y(s) \right] + (1 - q_i) (1 - q_j) E_{i,t}^{d,d} \left[ \sum_{s=t+1}^{\infty} \beta^{(s-t)} u'(y(s))y(s) \right]$$

$$= q_i q_j S_{i}^{h,h} + q_i (1 - q_j) S_{i}^{h,d} + (1 - q_i) q_j S_{i}^{d,h} + (1 - q_i) (1 - q_j) S_{i}^{d,d},$$

where

$$S_{i}^{m,n} = \frac{\theta (1 - \mu_{im}) + \mu_{jn} y_i + (1 + \theta) \mu_{im} \left( \frac{y_i}{y_j} \right)^\gamma y_j}{\theta (\theta + \mu_{im} + \mu_{jn})},$$

is the price of the tree under complete information when transition intensities $(\mu_{im}, \mu_{jn})$ are known for $\{m, n\} \in \{l, h\}^2$. In this case, the equilibrium price of the tree is a linear combination of equilibrium prices of the tree under complete information. We now show that a similar result holds when agents have heterogeneous beliefs.

**5.3. Reformulation of the problem**

For $(m, n) \in \{l, h\}^2$, define

$$U_{i,t}^{m,n}(c) = E_{i,t}^{m,n} \left[ \sum_{s=t}^{\infty} \beta^{(s-t)} u(c(s)) \right].$$

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Then, we write
\[
E_{i,t}^s \left[ \sum_{s=t}^{\infty} \beta(s-t) u(c(s)) \right] = q_{iat} q_{jat} U_{i,t}^{h,h}(c) + q_{iat}(1 - q_{jat}) U_{i,t}^{h,l}(c) + \\
+ (1 - q_{iat}) q_{jat} U_{i,t}^{l,h}(c) + (1 - q_{iat})(1 - q_{jat}) U_{i,t}^{l,l}(c)
\]
\[
= E_{i,t}^{l,l} \left[ \sum_{s=t}^{\infty} \beta(s-t) \left( \frac{\pi_{ija}(s)}{P_{ij}^{l,l}(s)} u(c_{ija}(s)) + \frac{\pi_{ija}(s)}{P_{ij}^{l,l}(s)} u(c_{ija}(s)) \right) \right] \tag{5.1}
\]
with
\[
\pi_{ikk}(s) = q_{ikt} q_{jkt} P_{ii}^{h,h}(s) + q_{ikt}(1 - q_{jkt}) P_{ii}^{h,l}(s) + (1 - q_{ikt}) q_{jkt} P_{ii}^{l,h}(s) + (1 - q_{ikt})(1 - q_{jkt}) P_{ii}^{l,l}(s)
\]
\[
\pi_{ijk}(s) = q_{ikt} q_{jkt} P_{ij}^{h,h}(s) + q_{ikt}(1 - q_{jkt}) P_{ij}^{h,l}(s) + (1 - q_{ikt}) q_{jkt} P_{ij}^{l,h}(s) + (1 - q_{ikt})(1 - q_{jkt}) P_{ij}^{l,l}(s),
\]
for \( k = a, b \) and where
\[
P_{ii}^{h,l}(s) = \frac{\mu_{ji} + \mu_{ih}(1 - \mu_{il} - \mu_{jh})(s-t)}{\mu_{ih} + \mu_{jl}},
\]
is the probability that being in state \( i \) at time \( t \) to be in state \( i \) at time \( s \) and given the fact that the intensity of switching from state \( i \) (\( j \)) into state \( j \) (\( i \)) is \( \mu_{ih} \) (\( \mu_{jl} \)). Notice that
\[
\pi_{ikk}(s) + \pi_{ijk}(s) = 1.
\]
Relationship (5.1) allows to re-express the conditional expectation of one agent as a conditional expectation where the Poisson intensities of switching are known by the use of the Radon Nikodym derivative theorem. A major implication of this reformulation is that the methodology used in the non-learning case can be applied.

### 5.3.1. Some properties of the beliefs \( \pi \)

**P1.** If \( \mu_{1h} - \mu_{1l} = \mu_{2h} - \mu_{2l} \), one can show that for all \( s > t \)
\[
\pi_{iia}(s) > \pi_{iib}(s) \text{ if only if }\]
\[
q_{ibt} > q_{iat} \quad (1 - q_{ibt})(1 - q_{jbt}) > (1 - q_{iat})(1 - q_{jat}) \]
\[
\text{if } 2A + B \geq 0 \text{ and } B \leq 0 \text{ then } B^2 - 4AC \leq 0,
\]
with
\[
A = (q_{ibt} q_{jbt} - q_{iat} q_{jat}) \mu_{ih}
\]
\[
B = (q_{ibt}(1 - q_{jbt}) - q_{iat}(1 - q_{jat})) \mu_{ih} + (q_{jbt}(1 - q_{ibt}) - q_{jat}(1 - q_{iat})) \mu_{il}
\]
\[
C = ((1 - q_{ibt})(1 - q_{jbt}) - (1 - q_{iat})(1 - q_{jat})) \mu_{il},
\]
and we also have
\[
\pi_{ija}(s) < \pi_{ijb}(s).
\]
5.4. Equilibrium asset price

As in the non-learning case, being in state $i$ with beliefs pairs $(q_a, q_b)$, the equilibrium price of the tree is given by

$$S_i(q_a, q_b) = \sum_{s=t+1}^{\infty} \beta^{s-t} \left[ \left( (w_a \pi_{iia}(s))^{\frac{1}{\gamma}} + (w_b \pi_{iib}(s))^{\frac{1}{\gamma}} \right) y_i + \left( (w_a \pi_{ija}(s))^{\frac{1}{\gamma}} + (w_b \pi_{ijb}(s))^{\frac{1}{\gamma}} \right) \left( \frac{y_i}{y_j} \right)^{\gamma} y_j \right].$$

Note that for all $(i,j)$ in $\{1, 2\}$, beliefs $\pi_{iia}$ and $\pi_{ijb}$ are independent from the coefficient of relative risk aversion $\gamma$. It is easy to realize that results obtained in the non-learning case apply.

Obviously, as time passes, since the two agents receive the same information, beliefs converge and the speculation phenomenon (if any) will be dampened.

TO BE COMPLETED

5.5. General Belief Case

In this section, we sketch how to solve the problem under the assumption that investors have arbitrary (but equivalent) priors about the transitions probabilities. Since beliefs are assumed to be equivalent, let $\xi_{a,b}$ denote the Radon-Nikodym derivative of beliefs $P_b$ with respect to beliefs $P_a$, i.e., $\xi_{a,b} = \frac{dP_b}{dP_a}$. It follows that we can write

$$E_{i,b,0}^i \left[ \sum_{s=0}^{\infty} \beta^s u_b(c_b(s)) \right] = \frac{1}{\xi_{a,b}(0)} E_{a,0}^i \left[ \sum_{s=0}^{\infty} \beta^s \xi_{a,b}(s) u_b(c_b(s)) \right] = \frac{1}{\xi_{a,b}(0)} \sum_{s=0}^{\infty} \beta^s \left( \pi_{iia}(s) \xi_{i,i,b}^i(s) u_b(c_{iib}(s)) + \pi_{ija}(s) \xi_{i,a,b}^j(s) u_b(c_{ijb}(s)) \right).$$

6. Conclusion

We develop a simple two person general equilibrium model and derive conditions on preferences and beliefs under which a speculation phenomenon can arise. For the sake of simplicity, we have considered the case when there are only two states of the world but the analysis can be straightforwardly extended to $n$ states. We obtain that beliefs disagreements are necessary for the speculation phenomenon to take place as well as a low level of risk aversion. More participants with heterogeneous beliefs can enhance the speculation phenomena. On the opposite, when agents are risk averse enough, they all try to get rid of the risky asset leading to a bear market and the equilibrium asset price can be lower than the minimum fundamental asset valuation.

Finally, we also incorporate learning allowing agents to revise their beliefs upon the arrival of new public information. Under the same conditions as passive expected utility
maximizers with fixed expectations, speculation can take place but will be dampened as time passes since agents’ beliefs will eventually converge.
7. Appendix

7.1. Appendix 1

Derivation of conditional probability \( P_{ij}(s) \). Note that

\[
P_{ii}(t + 1) = (1 - \mu_i)P_{ii}(t) + \mu_j(1 - P_{ii}(t)).
\]

This is a first order difference equation with initial condition \( P_{ii}(0) = 1 \). It is easy to check that the solution is

\[
P_{ii}(t) = \frac{\mu_j + \mu_i(1 - \mu_i - \mu_j)^t}{\mu_i + \mu_j}.
\]

**Conjecture Verification.** Simple algebra leads to

\[
S_2 + y_2 - (S_1 + y_1) = \frac{(1 + \theta)(\phi_{1a} - \phi_{2b})(y_2 - y_1)}{\theta(\theta + \phi_{1a} + \phi_{2b})} > 0,
\]

since \( \phi_{1a} > \phi_{2b} \) and \( y_2 > y_1 \). Then, for \( i = 1, 2 \), \( S_i > S_{ia} \) if and only if

\[
\frac{(\theta(1 - \phi_{1a}) + \phi_{2b})y_1 + (1 + \theta)\phi_{1a}y_2}{\theta(\theta + \phi_{1a} + \phi_{2b})} > \frac{(\theta(1 - \phi_{1a}) + \phi_{2a})y_1 + (1 + \theta)\phi_{1a}y_2}{\theta(\theta + \phi_{1a} + \phi_{2a})}
\]

and

\[
\frac{(1 + \theta)\phi_{2b}y_1 + (\theta(1 - \phi_{2b}) + \phi_{1a})y_2}{\theta(\theta + \phi_{1a} + \phi_{2b})} > \frac{(1 + \theta)\phi_{2a}y_1 + (\theta(1 - \phi_{2a}) + \phi_{1a})y_2}{\theta(\theta + \phi_{1a} + \phi_{2a})}.
\]

These two inequalities are satisfied exactly when \( y_2 > y_1 \) and \( \phi_{2a} > \phi_{2b} \), which are the assumptions we made. In the same fashion, since \( \phi_{1a} > \phi_{1b} \), we have

\[
S_i > S_{ib} \text{ for } i \in \{1, 2\}.
\]

**Existence and uniqueness of the equilibrium.** Let us define

\[
x = \frac{\frac{\lambda_a}{\lambda_a^2 + \lambda_b^2}}{\frac{\lambda_a}{\lambda_a^2 + \lambda_b^2}},
\]

and two new stochastic processes \( f \) and \( Y \) such that under the probability measure induced by agent \( a \) beliefs

\[
f(s) = \begin{cases} 
\left( \frac{p_{ib}(s)}{p_{ia}(s)} \right)^{\frac{1}{2}} & \text{in state } i \\
\left( \frac{p_{ja}(s)}{p_{jia}(s)} \right)^{\frac{1}{2}} & \text{in state } j
\end{cases}
\]

and

\[
Y(s) = \begin{cases} 
y_i \beta^s & \text{in state } i \\
y_j \beta^s & \text{in state } j
\end{cases}
\]
It can easily be checked that there is a one to one correspondence between $x$ and $\lambda_i$ and moreover $x$ must be solution of

$$
E^i_{a,0} \left[ \sum_{s=1}^{\infty} x (x + (1 - x)f(s))^{\gamma-1} Y(s) \right] - x_{ia}(0)
$$

or equivalently

$$
E^i_{a,0} \left[ \sum_{s=1}^{\infty} (x + (1 - x)f(s))^{\gamma} Y(s) \right] = z_{ia}^{-}(0),
$$

with $X = \frac{x}{1-x} \in \mathbb{R}_+$. Let us define an auxiliary function

$$
F : X \mapsto \frac{xG_{\gamma-1}(X)}{G_{\gamma}(X)},
$$

with

$$
G_{\gamma}(X) = E^i_{a,0} \left[ \sum_{s=1}^{\infty} (X + f(s))^{\gamma} Y(s) \right].
$$

$F$ is continuously differentiable with $F(0) = 0$ and $\lim_{X \to \infty} F(X) = 1$. Then

$$
F'(X) = \frac{G_{\gamma}(X) (G_{\gamma-1}(X) + (\gamma-1)XG_{\gamma-2}(X)) - \gamma XG_{\gamma-1}^2(X)}{G_{\gamma}^2(X)}.
$$

Since $G_{\gamma-1}(X) > XG_{\gamma-2}(X)$, in order to prove that $F'(X) > 0$, it is enough to show that

$$
G_{\gamma}(X)G_{\gamma-2}(X) > G_{\gamma-1}^2(X)
$$

or equivalently,

$$
E^i_{a,0} \left[ \sum_{s=1}^{\infty} (X + f(s))^{\gamma} Y(s) \right] E^i_{a,0} \left[ \sum_{s=1}^{\infty} (X + f(s))^{\gamma-2} Y(s) \right] > \left( E^i_{a,0} \left[ \sum_{s=1}^{\infty} (X + f(s))^{\gamma-1} Y(s) \right] \right)^2.
$$

Note that $Y$ is a positive stochastic process and

$$
S \times S \to \mathbb{R}_+,
$$

$$
<.,>: (U,V) \mapsto E^i_{a,0} \left[ \sum_{s=1}^{\infty} U(s)V(s)Y(s) \right],
$$

is an inner product on $S$, the set of integrable stochastic processes. Choosing $(U(s), V(s)) = \left( (X + f(s))^{\frac{\gamma-2}{2}} , (X + f(s))^{\frac{\gamma-2}{2}} \right)$ and using the Cauchy-Schwarz inequality, it follows that

$$
< U, V >^2 \leq < U, U > \times < V, V >,
$$

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with equality exactly when \( U = \psi V \) for some constant \( \psi \). In our case, \( U \) and \( V \) cannot be proportional, so the inequality is strict and we obtain that \( F' \) is positive. This implies that the equation \( F(X) = z_{ia} \) has a unique root. It follows that \( x = \frac{X}{1+rX} \) is uniquely defined and so is \( \lambda_o \). ■

**Interest rate.** Being in state \( i \), the equilibrium interest rate \( r_i \) must satisfy

\[
\frac{1}{1+r_i} u'(c_{ik}(t)) = \beta E_{k,t}^i [u'(c_{ik}(t+1))],
\]

for \( k \in \{a, b\} \). For instance, for \( k = a \), choosing \( i = 1 \), we have

\[
\frac{1}{1+r_i} \frac{\left(\lambda_a^\frac{1}{\gamma} + \lambda_b^\frac{1}{\gamma}\right)}{\lambda_a} y_i^{-\gamma} = \beta \left((1-\phi_{ia}) \frac{\left(\lambda_a^\frac{1}{\gamma} (1-\phi_{ia})^\frac{1}{\gamma} + \lambda_b^\frac{1}{\gamma} (1-\phi_{ib})^\frac{1}{\gamma}\right)}{\lambda_a(1-\phi_{ia})} \right) y_i^{-\gamma}
\]

\[
+ \phi_{ia} \frac{\left(\lambda_a^\frac{1}{\gamma} \phi_{ia}^\frac{1}{\gamma} + \lambda_b^\frac{1}{\gamma} \phi_{ib}^\frac{1}{\gamma}\right)}{\lambda_a \phi_{ia}} y_j^{-\gamma}.
\]

The desired result follows easily. ■

**7.2. Appendix 2**

**Proof of lemma 1.** For \( \gamma \geq 1 \), the function \( x \mapsto x^{\frac{1}{\gamma}} \) is concave. Writing

\[
\frac{(\lambda x)^\frac{1}{\gamma} + ((1-\lambda)y)^\frac{1}{\gamma})^\gamma}{(\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma})^\gamma} = \left(\frac{\lambda^\frac{1}{\gamma}}{\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma}} x^{\frac{1}{\gamma}} + \frac{(1-\lambda)^\frac{1}{\gamma}}{\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma}} y^{\frac{1}{\gamma}}\right)^\gamma
\]

by concavity we have

\[
\frac{(\lambda x)^\frac{1}{\gamma} + ((1-\lambda)y)^\frac{1}{\gamma}}{\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma}} \leq \left(\frac{\lambda^\frac{1}{\gamma}}{\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma}} x^{\frac{1}{\gamma}} + \frac{(1-\lambda)^\frac{1}{\gamma}}{\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma}} y^{\frac{1}{\gamma}}\right)^\frac{1}{\gamma}.
\]

This implies that

\[
\frac{(\lambda x)^\frac{1}{\gamma} + ((1-\lambda)y)^\frac{1}{\gamma})}{(\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma})^\gamma} \leq \frac{\lambda^\frac{1}{\gamma}}{\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma}} x + \frac{(1-\lambda)^\frac{1}{\gamma}}{\lambda^\frac{1}{\gamma} + (1-\lambda)^\frac{1}{\gamma}} y,
\]

with strict inequality when \( \gamma > 1 \) and \( x \neq y \). Conversely, when \( \gamma \leq 1 \), the function \( x \mapsto x^{\frac{1}{\gamma}} \) is convex, so the inequality is reversed. ■
Proof of lemma 2. For $\gamma$ in the neighborhood of 0, since $f$ and $g$ are continuous, using a first order Taylor expansion we can write

\begin{align*}
  f(\gamma) &= f(0)(1 + \alpha(\gamma)) \\
  g(\gamma) &= g(0)(1 + \beta(\gamma)),
\end{align*}

where

\[
  \lim_{\gamma \to 0} \alpha(\gamma) = \lim_{\gamma \to 0} \beta(\gamma) = 0.
\]

Without loss of generality, assume that $f(0) \geq g(0)$, then

\[
  \left( f(\gamma)^{\frac{1}{\gamma}} + g(\gamma)^{\frac{1}{\gamma}} \right)^\gamma = f(0) \left( (1 + \alpha(\gamma))^\frac{1}{\gamma} + \left( \frac{g(0)}{f(0)} (1 + \beta(\gamma))^\frac{1}{\gamma} \right) \right)^\gamma,
\]

and we want to show that

\[
  \lim_{\gamma \to 0} \left( (1 + \alpha(\gamma))^\frac{1}{\gamma} + \left( \frac{g(0)}{f(0)} (1 + \beta(\gamma))^\frac{1}{\gamma} \right) \right)^\gamma = 1.
\]

On the one hand, we have

\[
  1 + \alpha(\gamma) \leq (1 + \alpha(\gamma))^\frac{1}{\gamma} + \left( \frac{g(0)}{f(0)} (1 + \beta(\gamma))^\frac{1}{\gamma} \right)^\gamma,
\]

and on the other hand

\[
  \left( (1 + \alpha(\gamma))^\frac{1}{\gamma} + \left( \frac{g(0)}{f(0)} (1 + \beta(\gamma))^\frac{1}{\gamma} \right) \right) \leq \left( 2(1 + |\alpha(\gamma)| + |\beta(\gamma)|)^\frac{1}{\gamma} \right) \leq 2^\gamma (1 + |\alpha(\gamma)| + |\beta(\gamma)|).
\]

The desired result follows easily. Therefore $\lim_{\gamma \to 0} \left( f(\gamma)^{\frac{1}{\gamma}} + g(\gamma)^{\frac{1}{\gamma}} \right)^\gamma = f(0)$. 

Interest rate property. If $\lambda_a > \frac{1}{2}$ and $\lambda_a \phi_{ia} > \lambda_b \phi_{ib}$ and $\lambda_a(1 - \phi_{ia}) > \lambda_b(1 - \phi_{ib})$ then $r_i = \theta$. If $\lambda_a > \frac{1}{2}$ and $\lambda_a \phi_{ia} > \lambda_b \phi_{ib}$ and $\lambda_a(1 - \phi_{ia}) < \lambda_b(1 - \phi_{ib})$ then

\[
  \frac{1}{1 + r_i} = \beta \left[ \phi_{ia} + \frac{\lambda_b(1 - \phi_{ib})}{\lambda_a} \right],
\]

which implies that $r_i < \theta$. If $\lambda_a > \frac{1}{2}$ and $\lambda_a \phi_{ia} < \lambda_b \phi_{ib}$ and $\lambda_a(1 - \phi_{ia}) > \lambda_b(1 - \phi_{ib})$ then

\[
  \frac{1}{1 + r_i} = \beta \left[ \frac{\lambda_b \phi_{ib} + \lambda_b(1 - \phi_{ia})}{\lambda_a} \right],
\]

and again, $r_i < \theta$. Finally, $\lambda_a > \frac{1}{2}$ and $\lambda_a \phi_{ia} < \lambda_b \phi_{ib}$ and $\lambda_a(1 - \phi_{ia}) < \lambda_b(1 - \phi_{ib})$ then

\[
  \frac{1}{1 + r_i} = \beta \frac{\lambda_b}{\lambda_a},
\]

and again, $r_i < \theta$. Reversing the role of $\lambda_a$ and $\lambda_b$ shows that in all cases $r_i \leq \theta$, with possibly strict inequality.
7.3. Appendix 3

**Step 1.** Assume that \(z_1 < \frac{1}{2}\). Then, we postulate that \(\lambda \sim m^\gamma\), for some \(m < 1\) to be determined. The first step is to show that if \(\lambda \sim m^\gamma\), then for all \((u, v)\) in \(\mathbb{R}_+^2\):

\[
\lim_{\gamma \to \infty} \left( \frac{(\lambda u)^{\frac{1}{\gamma}} + ((1 - \lambda)v)^{\frac{1}{\gamma}}}{(\lambda^\frac{1}{\gamma} + (1 - \lambda)^{\frac{1}{\gamma}})^\gamma} \right) = u^{\frac{m}{1+m}} v^{\frac{1}{1+m}}.
\]  

(7.1)

We use a Taylor expansion to write

\[
\left( \frac{(\lambda u)^{\frac{1}{\gamma}} + ((1 - \lambda)v)^{\frac{1}{\gamma}}}{(\lambda^\frac{1}{\gamma} + (1 - \lambda)^{\frac{1}{\gamma}})^\gamma} \right) = e^{\gamma \ln \left( \frac{(\lambda u)^{\frac{1}{\gamma}} + ((1 - \lambda)v)^{\frac{1}{\gamma}}}{(\lambda^\frac{1}{\gamma} + (1 - \lambda)^{\frac{1}{\gamma}})^\gamma} \right)} = e^{\gamma \ln \left( e^{\frac{1}{\gamma} \ln \lambda u} + e^{\frac{1}{\gamma} \ln (1 - \lambda)v} \right)} = e^{\gamma \ln \left( (1 + \ln u) + 1 + \ln (1 - \lambda)v + o(\frac{1}{\gamma}) \right)} = (1 + m) e^{\frac{m \ln u + \ln (1 - \lambda)v + o(1)}{\gamma}}.
\]

Hence

\[
\left( \frac{(\lambda u)^{\frac{1}{\gamma}} + ((1 - \lambda)v)^{\frac{1}{\gamma}}}{(\lambda^\frac{1}{\gamma} + (1 - \lambda)^{\frac{1}{\gamma}})^\gamma} \right) = (1 + m) e^{\frac{m \ln u + \ln (1 - \lambda)v + o(1)}{\gamma}} = u^{\frac{m}{1+m}} v^{\frac{1}{1+m}} (1 + o(\frac{1}{\gamma})).
\]

The desired result follows. Then, since \(y_1 < y_2\) we have \(\lim_{\gamma \to \infty} \left( \frac{y_1}{y_2} \right)^\gamma = 0\). Finally

\[
\lim_{\gamma \to \infty} S_1 = \sum_{s=1}^{\infty} \beta^s \frac{\left( \left( \lambda_1 P_1(s) \right)^{\frac{1}{\gamma}} + \left( \lambda_2 P_2(s) \right)^{\frac{1}{\gamma}} \right)^\gamma}{\left( \lambda_1^\frac{1}{\gamma} + \lambda_2^\frac{1}{\gamma} \right)^\gamma} y_1 + \lim_{\gamma \to \infty} \frac{\left( \left( \lambda_3 P_3(s) \right)^{\frac{1}{\gamma}} + \left( \lambda_4 P_4(s) \right)^{\frac{1}{\gamma}} \right)^\gamma}{\left( \lambda_3^\frac{1}{\gamma} + \lambda_4^\frac{1}{\gamma} \right)^\gamma} \left( \frac{y_1}{y_2} \right)^\gamma y_2,
\]

where the introduction of the limit under the sum sign can be justified using Lebesgue dominated theorem. This leads to

\[
\lim_{\gamma \to \infty} S_1 = \left( \sum_{s=1}^{\infty} \beta^s \frac{m}{1+m} \beta^s P_{11a}(s) P_{11b}(s) \right) y_1.
\]
Given what precedes, it is easy to see that \( \lim_{\gamma \to \infty} S_2 = \infty \). It remains to identify \( m \) by using the budget constraint. First of all, note that
\[
\lim_{\gamma \to \infty} \left( \frac{\lambda_a p_{ij_a}(s)}{\lambda_b p_{ij_b}(s)} \right)^{\frac{1}{\gamma}} = \frac{m}{1 + m}.
\]
Hence since \( \lim_{\gamma \to \infty} \left( \frac{m}{y_1 y_2} \right)^{\gamma} = 0 \), \( m \) must satisfy
\[
\sum_{s=1}^{\infty} \beta^s p_{11a}^{\frac{m}{1+m}}(s) p_{11b}^{\frac{1}{1+m}}(s) y_1 = \frac{m}{1 + m} \left( \sum_{s=1}^{\infty} \beta^s p_{11a}(s) p_{11b}(s) \right),
\]
which yields
\[
z_{1a} = \frac{m}{1 + m},
\]
or
\[
m = \frac{z_{1a}}{1 - z_{1a}} < 1,
\]
since \( z_{1a} < \frac{1}{2} \).

**Step 2.** Let \( C^0[0,1] \) be the set of continuous functions defined on \( \mathbb{R}_+ \) onto \([0,1]\). Then, for \( \theta > 0 \), it is easy to check that
\[
C^0[0,1] \times C^0[0,1] \to \mathbb{R}_+ \quad <,> : (f, g) \mapsto \sum_{s=1}^{\infty} \beta^s f(s) g(s),
\]
is an inner product on \( C^0[0,1] \). Since \( z_{1a} \) and \( z_{1b} \) are conjugate numbers, i.e. \( z_{1a} + z_{1b} = 1 \), choosing \( f(s) = p_{11a}^{z_{1a}}(s) \) and \( g(s) = p_{11b}^{z_{1b}}(s) \) and using Hölder inequality, it follows that
\[
< f, g > \leq \left( < f^{\frac{1}{z_{1a}}, f^{\frac{1}{z_{1a}}}} > \right)^{z_{1a}} \times \left( < g^{\frac{1}{z_{1b}}, g^{\frac{1}{z_{1b}}}} > \right)^{z_{1b}},
\]
with equality exactly when \( f \equiv \psi g \) for some constant \( \psi \). In our case, \( p_{11a} \) and \( p_{11b} \) are proportional, so we have
\[
\left( \sum_{s=1}^{\infty} \beta^s p_{11a}^{z_{1a}}(s) p_{11b}^{z_{1b}}(s) \right)^{\frac{z_{1a}}{z_{1b}}} = \left( \sum_{s=1}^{\infty} \beta^s p_{11a}(s) \right)^{\frac{z_{1a}}{z_{1b}}} \times \left( \sum_{s=1}^{\infty} \beta^s p_{11b}(s) \right)^{\frac{z_{1b}}{z_{1b}}}.
\]

**Interest rate.** Given preliminary result (7.1), we have
\[
\lim_{\gamma \to \infty} \frac{1}{1 + r_1} = \lim_{\gamma \to \infty} \beta \left[ (w_a(1 - \phi_{1a})^{\frac{1}{\gamma}} + w_b(1 - \phi_{1b})^{\frac{1}{\gamma}})^\gamma + (w_a \phi_{1a}^{\frac{1}{\gamma}} + w_b \phi_{1b}^{\frac{1}{\gamma}})^\gamma \left( \frac{y_1}{y_2} \right)^\gamma \right]
\]
\[
= \beta \left[ \lim_{\gamma \to \infty} \left( w_a(1 - \phi_{1a})^{\frac{1}{\gamma}} + w_b(1 - \phi_{1b})^{\frac{1}{\gamma}} \right)^\gamma + \lim_{\gamma \to \infty} \left( w_a \phi_{1a}^{\frac{1}{\gamma}} + w_b \phi_{1b}^{\frac{1}{\gamma}} \right)^\gamma \left( \frac{y_1}{y_2} \right)^\gamma \right].
\]
Since $w_a \sim z_{1a}$, we have

$$
\lim_{\gamma \to \infty} \left( w_a (1 - \phi_{1a})^{\frac{1}{\gamma}} + w_b (1 - \phi_{1b})^{\frac{1}{\gamma}} \right)^\gamma = (1 - \phi_{1a})^{z_{1a}} (1 - \phi_{1b})^{z_{1b}},
$$

and

$$
\lim_{\gamma \to \infty} \left( \frac{y_1}{y_2} \right)^\gamma = 0. \text{ Hence}
$$

$$
\lim_{\gamma \to \infty} \frac{1}{1 + r_1} = \beta (1 - \phi_{1a})^{z_{1a}} (1 - \phi_{1b})^{z_{1b}}
$$

$$
\lim_{\gamma \to \infty} r_2 = -1.
$$

It is easy to see that

$$
\lim_{\gamma \to \infty} \frac{1 + r_{2a}}{1 + r_2} = \left( \frac{\phi_{2a}}{\phi_{2b}} \right)^{z_{1b}}
$$

$$
\lim_{\gamma \to \infty} \frac{1 + r_{2b}}{1 + r_2} = \left( \frac{\phi_{2a}}{\phi_{2b}} \right)^{z_{1a}}.
$$

Finally,

$$
\lim_{\gamma \to \infty} c_{ija} = \lim_{\gamma \to \infty} \frac{(\lambda_{aP_{ija}}(s))^{\frac{1}{\gamma}} y_j}{(\lambda_{aP_{ija}}(s))^{\frac{1}{\gamma}} + (\lambda_{bP_{ijb}}(s))^{\frac{1}{\gamma}}}
$$

$$
= z_{i} y_{j}.
$$

7.4. Appendix 4

**Proof.** Dropping the time index $t$, we have found that

$$
c_{ij}^a (\lambda_{bP_{ijb}})^{\frac{1}{\gamma}} + c_{ij} (\lambda_{aP_{ija}})^{\frac{1}{\gamma}} = (\lambda_{aP_{ija}})^{\frac{1}{\gamma}} y_j,
$$

Now assume that

$$
\exists \lim_{\gamma \to 0} c_{ija} = k_j \in [0, y_j).
$$

Since $k_j < y_j$, it follows that

$$
c_{ij}^a (\lambda_{bP_{ijb}})^{\frac{1}{\gamma}} \sim 0 (\lambda_{aP_{ija}})^{\frac{1}{\gamma}} (y_j - k_j),
$$

or equivalently

$$
c_{ij}^\alpha \sim 0 \frac{\lambda_{bP_{ijb}}}{\lambda_{aP_{ija}}} (y_j - k_j)^{-\gamma}
$$

$$
\sim 0 \frac{\lambda_{bP_{ijb}}}{\lambda_{aP_{ija}}}.
$$

Hence

$$
\lim_{\gamma \to 0} c_{ij}^\alpha = \frac{\lambda_{bP_{ijb}}}{\lambda_{aP_{ija}}}.
$$
Since we must have \( c_{ija} \leq y_j \), this implies

\[
\frac{\lambda_a p_{ija}}{\lambda_b p_{ijb}} \leq y_j^\alpha.
\]  

(7.2)

Now assume that

\[
\lim_{\gamma \to 0} c_{ija} = y_j,
\]

then we must have

\[
\lim_{\gamma \to 0} \frac{a}{c_{ija}^{\gamma}} \left( \frac{\lambda_b p_{ijb}}{\lambda_a p_{ija}} \right)^{\frac{1}{\alpha}} = 0,
\]

which in turn implies

\[
\frac{\lambda_b p_{ijb}}{\lambda_a p_{ija}} y_j^{\alpha} < 1,
\]  

(7.3)

or equivalently

\[
y_j^{\alpha} < \frac{\lambda_a p_{ija}}{\lambda_b p_{ijb}}.
\]

Combining conditions (7.2) and (7.3) with what precedes yields

\[
\lim_{\gamma \to 0} c_{ija} = \min_{a.s.} \{ y_j, \left( \frac{\lambda_a p_{ija}}{\lambda_b p_{ijb}} \right)^{\frac{1}{\alpha}} \}
\]

\[
\lim_{\gamma \to 0} p_{ija} = \max_{a.s.} \left\{ \frac{\lambda_b p_{ijb}}{\lambda_a}, p_{ija} y_j^{-\alpha} \right\}.
\]

7.5. Appendix 5

**Proof.** Being in state \( i \) at time \( t \) and remaining in state \( i \) at time \( t + 1 \), applying Bayes rules leads to

\[
q_{ik}(t + 1) = \frac{q_{ik}(t)(1 - \mu_{ih})}{q_{ik}(t)(1 - \mu_{ih}) + (1 - q_{ik}(t))(1 - \mu_{il})}.
\]

In the same way, being in state \( i \) at time \( t \) and switching to state \( j \) at time \( t + 1 \), using Bayes rules leads to

\[
q_{ik}(t + 1) = \frac{q_{ik}(t)\mu_{ih}}{q_{ik}(t)\mu_{ih} + (1 - q_{ik}(t))\mu_{il}}.
\]
8. References


