Anonymity in Large Societies*

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Abstract

In a social choice model with an infinite number of agents, there may occur "equal size" coalitions that a preference aggregation rule should treat in the same manner. We introduce an axiom of equal treatment with respect to a measure of coalition size and explore its interaction with common axioms of social choice. We show that, provided the measure space is sufficiently rich in coalitions of the same measure, the new axiom is the natural extension of the concept of anonymity, and in particular plays a similar role in the characterization of preference aggregation rules. *JEL* D71, C69

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1 Introduction

It has long been known that social choice models with an infinite number of agents differ in significant ways from models with finitely many agents. In particular, large societies admit preference aggregation rules satisfying the standard Arrovian axioms of efficiency, independence, non-dictatorship and transitivity (see, for instance, Fishburn [12], Kirman and Sondermann [15] or Hansson [13]). These Arrovian rules involve, in the words of Kirman and Sondermann [15], "invisible dictators:" collections of decisive coalitions that recede into infinity so that no particular individual has to be a member of every decisive coalition. Kirman and Sondermann showed that if there is a finite, countably additive measure on the agent space then for any Arrovian rule there are arbitrarily small decisive coalitions. However, Schmitz [24] showed that when the measure on the agent space is allowed to be infinite then there are Arrovian rules for which all decisive coalitions have measure ∞. Armstrong [2] extended the earlier results to societies where the set of admissible coalitions is restricted to be an algebra, with the implied measurability restriction on the social choice rules.

One intuitively appealing way to reestablish Arrow's impossibility result in the context of large societies is to show that Arrovian rules violate some form of equal treatment. Surprisingly, it seems that no attempt has been made to tie the measure of coalition size with the idea of equal treatment. Instead, Mihara [20] recently extended the usual notion of anonymity from a finite-agent to the infinite-agent set-up. The usual anonymity axiom states that a social choice rule should be insensitive to permutations of the agent space. Mihara [20] found that, subject to a simple richness condition on the algebra of admissible coalitions, any measurable Arrovian rule must treat differently coalitions of the same cardinality and thus violates anonymity. However, whenever the set of agents is endowed with a measure-space structure, there is no reason why a cardinality-based equal-treatment notion should be the relevant one. In fact, in certain applications it may be rather hard to justify, since it would require equal treatment of intuitively very different coalitions such as those including, respectively, every second and every thousandth agent.

A number of restrictions on agent permutations allowed in the definition of anonymity have therefore been proposed. One of these, which seems to avoid some of the counterintuitive implications mentioned above, has been proposed by Lauwers [16, 17, 18]. This is "bounded anonymity," which, as has been shown by Lauwers [17, 18], can be viewed as requiring admissible permutations to be measure-preserving, for most subsets of the agent space, for a particular class of ("density") measures on the set of agents. This last anonymity notion has been used by Fey [11] to extend May's [19] characterization of the majority rule to an infinite-agent setting. Our framework allows us to define anonymity in a general way that largely encompasses this earlier approach. In particular, we define conditions on the measure spaces of agents under which our anonymity notion acts in the same manner as the usual anonymity axiom in a finite-agent setting, and we provide examples of measures of economic interest (including, but not limited to, the density measures of Lauwers [17]), that satisfy these assumptions.

We study the consequences of explicitly incorporating the notion of the coalition measure size into a social choice model and requiring equal-treatment of coalitions of equal measure. We offer three sets of results. First, we show that, subject to a richness condition on the equal-measure coalition classes, equal-treatment results in the restoration of Arrow's impossibility. Second, we drop the requirement of transitivity of social indifference, and show that, with this adjustment, our equal treatment axiom is consistent with Pareto and independence of irrelevant alternatives. Subject to another richness condition on the equal-measure coalition classes, the only preference aggregation rules satisfying Pareto, equal-treatment, neutrality and quasi-transitivity are the *consensual rules*: rules that give veto power to every positive measure coalition. Third, we show that Pareto, equal-treatment, neutrality and monotonicity characterize a class of preference aggregation rules which we call *measuring rules*. These rules are the large society equivalent to the counting rules described by Austen-Smith and Banks [6].

Recently, Banks, Duggan and Le Breton [7] have extended the theory of the core, the uncovered set, and the related undominated set to a model with a general set of alternatives and a measure space of voters, showing, inter alia, the general emptiness of the core and the general non-emptiness of the uncovered and the undominated set. The anonymity property that they define in the context of simple games corresponds to our equal treatment notion. The focus of their work, however, is quite different.

It should be noted that our approach to modelling large societies is entirely consistent with the now standard approach to modelling large competitive economies in general equilibrium theory (see Hildenbrand [14]). Given the objectives of this study, we do not require many of the usual assumptions in that literature. However, care is taken to ensure that all of our results go through in that setting.

The rest of this paper is organized as follows. In section 2 we present a model of measurable social choice following Armstrong [2]. In section 3 we introduce the notion of coalition size. In section 4 we define the property of anonymity with respect to a measure and explore the consequences of imposing it in the place of the standard cardinality-based anonymity property. In Section 5 we discuss possible extensions.

2 Social Choice on Coalition Algebras

2.1 Agents and Coalitions

Let N be the (non-empty) set of individuals (voters); it can be either a finite or a (countably or uncountably) infinite set. A *coalition* is any subset $L \subset N$. A *coalition algebra* is any class of coalitions $\mathcal{L} \subset 2^N$ that contains N itself and is closed under the formation of complements and finite unions (or, equivalently, finite intersections). The pair $(N, \mathcal{L})$ is a *coalition space*.

A coalition algebra is a collection of admissible coalitions that satisfies some minimum requirements: the union of two admissible coalitions should itself be admissible, and the complement of an admissible coalition should also be admissible. The admissibility restriction may arise from the nature of the economic model at hand; alternatively, it could be viewed as coming from observability or computability constraints facing the social planner as in [21]. The fact that the set of admissible coalitions is required to be an algebra reflects our interest in being able to make statements like: "the set of people not in coalition A" or "the set of those either in A or in B." The results that follow do not require that the set of coalitions form a σ-algebra; many interesting examples with a countably infinite set of voters would fail that requirement. On the other hand, there are many models, such as the standard treatment of a competitive economy with a continuum of agents (following Aumann [5]), where the set of admissible coalitions is a σ-algebra. Naturally, our results apply in such settings as well.

We next consider coalition spaces that are fine enough to admit all coalitions of interest in some examples of large societies.

2.2 Some Coalition Spaces

The usual social choice framework considers a finite set N of individuals. In this case, any coalition is considered admissible, so $\mathcal{L} = 2^N$.

For any N infinite, consider the algebra $\mathcal{L}_{cN}$ which is composed of all finite (including the empty set) and cofinite (complements of finite) subsets of N. This is the coarsest algebra that admits all the singleton coalitions, so, in a sense, it is the coarsest algebra of interest. We call $(N, \mathcal{L}_{cN})$ the *cofinite coalition space.*

In the context of countable societies, some finer algebras may be of interest, as suggested by the following example proposed by Mihara [20]. Consider a society composed of finitely many people, where there is uncertainty expressed as a countable number of states of the world. Index the persons by $j = 1, \ldots, n$ and the states by $s = 1, 2, \ldots$; we can let the preferences of person j in state s be represented by an individual $i = n(s-1) + j$. It seems natural to consider as admissible coalitions the sets $\{i \in \mathbb{N} : i = n\,(s-1) + j, s = 1, 2, \ldots\}$ (n-period sets) for $j = 1, \ldots, n$, representing the person j in all states of the world, and the sets $\{i \in \mathbb{N} : (s-1)\,n < i \leq s\,n\}$ for $s = 1, 2, \ldots$, representing all the individuals in a particular state. Note that an algebra containing both types of sets will include also all the coalitions consisting of a single individual, and hence all finite and cofinite coalitions. We denote this algebra as $\mathcal{L}_{p,n}$, and we call $(\mathbb{N}, \mathcal{L}_{p,n})$ the *n-period coalition space.*

We may also wish to consider a coalition algebra that recognizes all n-period sets for all n. The coarsest such algebra admits all finite unions of $n-$period sets (periodic sets), as well as all unions and differences of a periodic and a finite set. We denote this algebra by $\mathcal{L}_p$, and refer to $(\mathbb{N}, \mathcal{L}_p)$ as the *periodic coalition space.*

Another example where the $n-$period coalition space may be reasonable is the following. Consider a society composed of finitely many dynasties, which we index by $j = 1, \ldots, n$; at each period $s = 1, 2, \ldots$ each dynasty is represented by one individual. We can name an individual belonging to dynasty j and living in period s by $i = n(s-1) + j$. It seems natural to consider as admissible coalitions the sets including all members of a dynasty as well as the sets including all individuals living at a given period. Other coalition algebras on the same set of agents may be of interest as well. Of course, the finest such algebra is the power set 2^N of N. A coalition here is any collection of individuals.

As noted by Banks, Duggan, and Le Breton [7], uncountable societies, in the form of a continuous distribution of voters, appear in many examples in the political science literature, including the party competition model of Downs [9]. This is also the standard setting of the Aumann's [5] continuum of agents model of perfect competition. In this case it may be natural to consider the agent space to be $N = [0, 1]$ with the Borel σ-algebra of coalitions over it. A coalition here is a (Borel-measurable) collection of agents.

2.3 Measurable Preference Aggregation

In this section we set up the choice space of the problem and the space of agents' characteristics. We depart from the literature stemming from Kirman and Sondermann [15] by requiring the latter to be measurable with respect to the given coalition algebra. This is necessary given our choice of the equal-treatment notion that follows. Our treatment of the large society in this section follows Hildenbrand's [14] reformulation of Aumann's [5] model (see also Ellickson [10]). In this, our set-up is similar to Banks, Duggan and Le Breton [7]. At the same time, for the problem at hand we do not need all the assumptions on the structure of both the agents' characteristics' space and the space of alternatives that are imposed in the general equilibrium literature; in particular, in contrast to Hildenbrand [14] we drop the imposition of a topology upon the set of preferences.

Let X be a (finite or infinite) set of alternatives, which has at least three elements. Each individual has a preference relation (a reflexive, complete, and transitive binary relation, i.e. a weak order) on X. When it is convenient, we can view any binary relation (and in particular a preference relation) as a subset of the product space $X \times X$. We will usually denote a preference relation by the letter R; P will denote the corresponding strict preference (the asymmetric part of R), and I the indifference relation (the symmetric part of R). Denote by $\mathcal{R}$ the set of all weak orders on X.

Whenever X is a finite set, all preference relations are considered admissible. However, when X is infinite some restrictions must be imposed, like the consideration of only continuous preferences in general equilibrium models. Actually, we impose a constraint not on individual preferences, but on the *preference profiles*. In order to do this, we consider an algebra $\mathcal{E}$ on the space of preferences, so that the admissible preferences form a *type space* $(\mathcal{R}, \mathcal{E})$. Then, a preference profile is just a mapping from the coalition space of agents to the type space of preferences.

As noted above, we need not completely specify the choice of a topology on the type space $(\mathcal{R}, \mathcal{E})$ and derive from it the algebra on $\mathcal{R}$. However, since we are going to be interested in pairwise comparisons of alternatives we shall assume throughout the paper that $\mathcal{E}$ is sufficiently rich:

Assumption (UD) For any $x, y \in X$ the set $\{R \in \mathcal{R} : xRy\}$ belongs to $\mathcal{E}$.

We have termed this assumption (UD) because it stands for *Universal Domain*. Whenever X and N are finite, a basic assumption of Arrow's Theorem is the Universal Domain of preferences, *i.e.* all preference rankings are considered admissible. Our assumption (UD) is equivalent to the requirement that the sets of all weak orders which share an arbitrary ranking among *any finite set of alternatives* are admissible, and is precisely what is needed in order to ensure that the admissible preference profiles are rich enough so that the natural extensions of Arrow's Theorem continue to be valid.

In practice, the choice of $\mathcal{E}$ will derive from the economic and/or mathematical structure of the particular model at hand. In much of the existing social choice literature, where measurability issues do not explicitly arise, it is implicit that $\mathcal{E} = 2^{\mathcal{R}}$. On the other hand, the choice of the Borel σ-algebra derived from the topology of closed convergence, as in Hildenbrand [14], also satisfies assumption (UD) (see Banks, Duggan, and Le Breton [7]).

As we mentioned above, a *preference profile* is a mapping from the agent space to the type space

$$\rho : N \to \mathcal{R}$$

that assigns a preference relation $\rho(i) = R_i$ to each agent i (with P_i and I_i, respectively, its asymmetric and symmetric parts). We shall say that the profile ρ is measurable if for any $E \in \mathcal{E}$, the coalition $\{i : R_i \in E\}$ of all individuals whose preferences lie in E belongs to $\mathcal{L}$. All the preference profiles in what follows shall be assumed to be measurable.

Denote by $\mathcal{R}^N_{\mathcal{L},\mathcal{E}}$ the set of all (measurable) preference profiles, and by $\mathcal{B}$ the set of all reflexive and complete (not necessarily transitive) binary relations on X.

Definition 1 *A preference aggregation rule* is a map

$$f : \mathcal{R}^N_{\mathcal{L},\mathcal{E}} \to \mathcal{B}.$$

A preference aggregation rule assigns to every preference profile ρ a reflexive and complete binary relation, the *social* preference $R = f(\rho)$. We denote by P and I, respectively, the asymmetric and the symmetric parts of the social preference R.

A coalition $L \in \mathcal{L}$ is *decisive* under an aggregation rule f if

$$[\,\forall i \in L,\ xP_iy\,] \Rightarrow xPy$$

for all pairs $(x, y) \in X \times X$, and for all preference profiles.

Given a preference aggregation rule f, let $\mathcal{D}_f$ be the set of its associated decisive coalitions, and assume this set is not empty. Define a new preference aggregation rule $f_{\mathcal{D}_f}$ by

$$xPy \iff [\exists L \in \mathcal{D}_f : \forall i \in L,\ xP_iy]$$

for all pairs $(x, y) \in X \times X$, and for all preference profiles. We say that f is a *simple rule* if $f = f_{\mathcal{D}_f}$.

The following are some desirable criteria an aggregation rule might satisfy:

Definition 2 A preference aggregation rule f is

(P1) *weakly Paretian* if, for every $\rho \in \mathcal{R}^N_{\mathcal{L},\mathcal{E}}$ and for any $x, y \in X$,

$$[\,\{i \in N : xP_iy\} = N\,] \Rightarrow xPy.$$

(P2) *independent of irrelevant alternatives (IIA)* if for every $\rho, \rho' \in \mathcal{R}^N_{\mathcal{L},\mathcal{E}}$ and for any $x, y \in X$,

$$[\,\rho|_{\{x,y\}} = \rho'|_{\{x,y\}}\,] \;\Rightarrow\; [\,f(\rho)|_{\{x,y\}} = f(\rho')|_{\{x,y\}}\,],$$

where $\rho|_S$ represents the restriction of ρ to the set S.

(P3) *nondictatorial* if there does not exist $i \in N$ such that xP_iy implies xPy for every $\rho \in \mathcal{R}^N_{\mathcal{L},\mathcal{E}}$ and for any $x, y \in X$.

(P4) *transitive* if for every $\rho \in \mathcal{R}^N_{\mathcal{L},\mathcal{E}}$ and for all $x, y, z \in X$,

$$xRy \;\&\; yRz \;\Rightarrow\; xRz.$$

Arrow's [4] impossibility theorem shows that, if N and X are finite and assumption (UD) is satisfied, there is no preference aggregation function satisfying properties (P1) to (P4). Fishburn [12], Kirman and Sondermann [15] and others have shown that, in fact, such functions are possible once an infinite set of voters is considered. Such Arrovian rules are, essentially, characterizable as limits of sets of decisive coalitions. As pointed out by Kirman and Sondermann [15] they could be viewed as *"invisible dictators"* (they would become actual dictators if the agent space is redefined as the Stone-Cěch compactification of N).

Armstrong [2] extended this result to general coalition spaces. Our first result is a straightforward corollary of Proposition 3.2 in Armstrong [2].

Whenever N is infinite, for any coalition space $(N, \mathcal{L})$ such that $\mathcal{L} \supset \mathcal{L}_{cN}$ (that is, such that $\mathcal{L}$ contains all singleton coalitions), define a preference aggregation rule σ_c by

$$xPy \iff [\{i \in N : xP_iy\} \text{ is a cofinite set}]$$

for all directed pairs $x, y \in X$ (the completeness requirement in the definition of preference aggregation rule, of course, implies that when the set $\{i \in N : xP_iy\}$ is infinite but not cofinite the rule would be indifferent between x and y). That is, σ_c is the simple rule with the cofinite sets as decisive coalitions. We say that a rule f is an *extension* of σ_c if $\mathcal{D}_f \supset \{L \in 2^N : L \text{ is cofinite}\}$.

Proposition 1 *Let N be an infinite set, and let assumption* (UD) *hold. For any coalition space $(N, \mathcal{L})$ such that $\mathcal{L} \supset \mathcal{L}_{cN}$, there exists at least one preference aggregation rule satisfying weak Paretianism* (P1)*, IIA* (P2)*, non-dictatorship* (P3) *and transitivity* (P4)*; any such rule is an extension of σ_c.*

PROOF: It follows immediately from Armstrong's [2] proposition 3.2 (as amended in [3]) that for every free ultrafilter $\mathcal{U}$ of coalitions in $\mathcal{L}$ there exists an aggregation rule satisfying (P1)-(P4) such that all coalitions in $\mathcal{U}$ are decisive, and vice-versa, for every measurable social choice rule satisfying (P1)-(P4) the set of decisive coalitions is a free ultrafilter.

It is easy to see that in every coalition space $(N, \mathcal{L})$ such that $\mathcal{L} \supset \mathcal{L}_{cN}$, the set of cofinite coalitions is a free filter. By an application of Zorn's lemma ([1], p. 32), there exists a free ultrafilter of coalitions in $\mathcal{L}$ that contains the set of cofinite coalitions.

Now suppose there exists a preference aggregation rule f satisfying (P1)-(P4), and a cofinite coalition $L_c \in \mathcal{L}$ that is not decisive under f. Since for every set in $\mathcal{L}$ either itself or its complement must be in the ultrafilter, L_c's complement $L_c^c \in \mathcal{U}$ is a finite decisive coalition under f. But (P3) implies that no individual is in every decisive coalition. That is, for every $i \in L_c^c$ there exists $L_i \in \mathcal{U}$ such that $i \notin L_i$. But then $L_c^c \cap \left(\cap_{i \in L_c^c} L_i\right) = \emptyset$. Hence, there exists a finite set of elements of $\mathcal{U}$ with an empty intersection, which contradicts the definition of a filter. $\square$

In general, explicitly constructing Arrovian rules may be rather difficult (see Mihara [22]). The problem becomes much easier once the set of admissible coalitions is restricted, as shown in a construction by Mihara [23]. For an easy example, consider a society consisting of two infinitely-lived dynasties. As discussed above, a natural algebra of admissible coalitions for such a society could be $\mathcal{L}_{p,2}$. The one-dynasty coalitions in this setting are represented by the sets of odd and even numbers, respectively. A preference aggregation rule satisfying all the four properties can be defined as follows:

$$xPy \Longleftrightarrow [\{j \in \mathbb{N} : xP_{2j}y\} \text{ is a cofinite set}]$$

for all directed pairs $x, y \in X$. In words, this rule says that an alternative is preferred to another if and only if all but finitely many members of the second coalition agree. It is obvious that all cofinite coalitions are decisive (notice, however, that σ_c itself is not Arrovian if $\mathcal{L} \neq \mathcal{L}_{cN}$, since it would violate transitivity).

A striking feature of the above example is that the social choice rule discriminates among dynasties: the "evens" eventually rule. In fact, as we show below, this has to be generally the case for rules satisfying (P1)-(P4). This failure to discriminate among dynasties seems to violate some notion of "equal treatment" of what should be "equals." To formalize this idea, however, we need to introduce a notion of coalition size.

3 Coalition Measures

3.1 Coalition Size

In a finite world, a coalition's size is easy to define as its cardinality. This is the idea behind the standard anonymity axiom in social choice, which says that a social choice rule should be invariant under the agents' permutations. Mihara [20] has shown that, subject to a richness condition on the algebra of admissible coalitions, invariance under permutations of agents is inconsistent with (P1) - (P4). Unfortunately, in an infinite society such an axiom may be hard to justify. Consider for instance the dynastic society example. It is straightforward to show that, if the number of dynasties n is larger than

2, there exists a permutation of the agent space that transforms a single dynasty into its union with another dynasty. It seems rather hard to insist on equal treatment of such clearly different coalitions.

We could, of course, recall that our dynastic society is endowed with an additional structure described by the coalition algebra $\mathcal{L}$. A natural question to ask is, therefore, if it alone can help resolve this problem. Unfortunately, referring to the example above, it is possible to construct periodically measurable permutations that change the period of a coalition. To sum up, while cardinality-based anonymity notions are undoubtedly of interest, in order to discuss coalition size in important applications they may be inadequate. We therefore proceed to define explicitly coalition size as its measure, and to study the consequences of requiring equal treatment of equal measure (rather than equal cardinality) coalitions.

Even if the set of individuals N is finite, the introduction of a measure allows the consideration of situations in which individuals (or institutions) may have different weights.

Definition 3 Given a coalition space $(N, \mathcal{L})$, a set function μ on the algebra $\mathcal{L}$ is a *coalition measure* if:

(i) $\mu(L) \in [0, 1]$ for $L \in \mathcal{L}$;

(ii) $\mu(\emptyset) = 0$ and $\mu(N) = 1$;

(iii) If $L_1, \ldots, L_n$ are disjoint $\mathcal{L}$-coalitions, then

$$\mu\left(\bigcup_{k=1}^{n} L_k\right) = \sum_{k=1}^{n} \mu(L_k).$$

Note that we only require the probability measure μ to be *finitely additive*.

The triple $(N, \mathcal{L}, \mu)$ is a *coalition measure space*. An *extension* of a measure space $(N, \mathcal{L}, \mu)$ is any measure space $(N, \mathcal{L}', \mu')$ such that $\mathcal{L}' \supset \mathcal{L}$ and $\mu'(L) = \mu(L)$ for every $L \in \mathcal{L}$.

3.2 Some Coalition Measure Spaces

We present here some coalition measure spaces of interest. Consider first the cofinite measurable space, and define the measure μ_c by

$$\mu_c(L) = \begin{cases} 0 & \text{if } L \text{ is finite} \\ 1 & \text{if } L \text{ is cofinite} \end{cases}$$

(note that this is the only coalition measure that assigns equal weight to all singleton coalitions in the cofinite coalition space). We refer to $(N, \mathcal{L}_{cN}, \mu_c)$ as the *cofinite coalition measure space*.

For another example, consider the n-period coalition measurable space $(\mathbb{N}, \mathcal{L}_{p,n})$. Treating all $n-$period coalitions as having equal size implies the following coalition measure on $\mathcal{L}_{p,n}$:

$$\mu_{p,n}(L) = \lim_{k\to\infty} \frac{1}{k} \# [m \in L : 1 < m \leq k].$$

This measure assigns 0 to every finite set, $\frac{1}{n}$ to every n-period set, and 1 to every cofinite set. We refer to $(\mathbb{N}, \mathcal{L}_{p,n}, \mu_{p,n})$ as the *n-period measure space*.

Consider the periodic coalition space $(\mathbb{N}, \mathcal{L}_p)$. Extending $\mu_{p,n}$ leads to the following measure defined on $\mathcal{L}_p$:

$$\mu_p(L) = \lim_{k\to\infty} \frac{1}{k} \#[m \in L : 1 < m \leq k].$$

Like $\mu_{p,n}$, the measure μ_p is *not* countably additive. Consider the following example. Let L_1 be the periodic set containing all individuals named with odd natural numbers. For each $n \geq 2$, define $L_n = L_1 \setminus \{k \in \mathbb{N} : 1 \leq k \leq n\}$. Then $\forall n$, $L_{n+1} \subset L_n$. Now $L_n \downarrow \emptyset$, since, given any $k \in \mathbb{N}$, $n > k \Rightarrow k \notin L_n$. But $\mu_p(L_n) \to 1/2$, since for all n, $\mu_p(L_n) = 1/2$. The reason why μ_p is not countably additive is that μ_p is very different from the regular probability measures on complete metric spaces. All such probability measures are tight, that is, most of their mass is concentrated on a compact set. On the other hand, as our previous example illustrates, the value of μ_p is independent of what happens in any finite set, that is, μ_p concentrates most of its mass "at infinity."

We refer to $(\mathbb{N}, \mathcal{L}_p, \mu_p)$ as the *periodic coalition measure space*. Obviously, the periodic measure space is an extension of every n-period measure space. Billingsley ([8], p. 577) presents an example showing that $(\mathbb{N}, \mathcal{L}_p, \mu_p)$ cannot be *uniquely* extended to include all sets such that μ_p is well-defined. However, as discussed in Lauwers [17], free ultrafilters on $\mathbb{N}$ can, in fact, be used to construct extensions μ_p to all subsets of $\mathbb{N}$. Naturally, since there are infinitely many such ultrafilters, there is no unique way of doing such an extension.

Going back to the dynastic example, we may think that the periodic measures are not the only ones of interest. We may wish to construct a measure that "discounts" the welfare of future generations according to a factor

$\beta \in (0, 1)$; in this case, we can construct a probability measure over $\mathcal{L}_{p,n}$ that assigns measure $1/n$ to every n-period set, and measure $(1-\beta)\beta^{s-1}/n$ to every individual j such that $n(s-1) < j \leq ns$. We refer to this as the (n, β)-*discounting measure space*. In the dynastic example may think of the n-period measure space as appropriate from a normative perspective if discounting of future generations is disallowed.

The discounting measure space admits atoms – in fact, it is purely atomic. Other reasons to allow atoms include making explicit the idea that some individuals (we could call them "politicians") may have a non-negligible weight even in a large society, whereas others may be negligible on their own. Alternatively, atoms may arise if we want to accommodate political organizations as "indivisible" coalitions, while also allowing for the presence of unorganized agents.

Finally, in the example with a continuous distribution of voters over the interval $[0, 1]$, a natural choice is the Lebesgue measure.

4 Equal-Treatment of Equal-Size Coalitions

4.1 Arrow Theorem Revisited

We now explore the consequences of adopting the concept of anonymity as an equal-treatment notion. The intuitive notion of anonymity is that the aggregation rule should not depend on the identities of the agents. In a finite setting, this requirement has been translated as the property that a permutation σ of the set of agents should not change the aggregation rule, that is, $f(\rho \circ \sigma) = f(\rho)$. Now, when N is infinite this definition must be generalized, and several (different) attempts have been made in the literature. Our setting allows a precise generalization, since we take as a departing point a *coalition measure space*. The problem in previous generalizations is that the finite case has an implicit measure (which assigns to each set its cardinality), and with an infinite N without a measure it is difficult to generalize the notion of permutations. To motivate our definition as the suitable generalization, let us go back to the previous definition in the finite case. Notice that the profile ρ and the profile $\rho \circ \sigma$ induce exactly *the same distribution on preferences*: each preference in the image is chosen as many times under ρ as under $\rho \circ \sigma$. If we look at the literature on large economies (see Hildenbrand [14] and Ellickson [10]), the justification for the *distributional approach* is precisely the fact that only the distribution of preferences and endowments should matter, not the identities of the individual agents. This expresses very accurately our notion of anonymity, which generalizes the finite one and is based on the distributions induced on preferences.

Given a coalition measure space $(N, \mathcal{L}, \mu)$, every (measurable) preference profile $\rho : N \to \mathcal{R}$ induces a (finitely additive) probability measure ν on the type space $(\mathcal{R}, \mathcal{E})$ defined by:

$$\nu(E) = \mu\left[\rho^{-1}(E)\right] = \mu\left\{i \in N : R_i \in E\right\}, \quad \text{for every } E \in \mathcal{E}.$$

This induced measure is usually denoted in probability theory as $\rho(\mu)$. We will use this notation. Now we are equipped to provide our definition of anonymity, which requires that the aggregation rule depends only on the distribution induced by the profile on the set of preferences.

Definition 4 Given a measure space $(N, \mathcal{L}, \mu)$, a preference aggregation rule f is

(P5) *μ-anonymous* if $\rho(\mu) = \rho'(\mu)$ implies $f(\rho) = f(\rho')$.

One implication of this definition is that sets of zero measure do not matter, since they do not alter the induced distribution on types. In particular, a zero–measure coalition can never be decisive. This, of course, means that whenever any single agent is negligible with respect to μ, μ-anonymity implies non-dictatorship (non-dictatorship is also implied, for instance, if every agent has the same measure).

This anonymity notion is closely related to the notion of anonymity among winning coalitions in simple games defined in a somewhat different setting by Banks, Duggan and Le Breton [7]. Note that the above requirement is considerably stronger than asking for equal-treatment of profiles that differ over a measure zero of agents. Our equal-treatment notion, however, has no "bite" unless the equal-measure coalition classes are sufficiently large. A particularly nasty example is the $(1, \beta)$-discounting measure space with $\beta < 1/2$. In this space there are no two equal-measure coalitions. This and similar examples can be avoided if the following richness condition on the coalition measure space is satisfied:

Assumption (R1): For every ultrafilter $\mathcal{U}$ on the coalition algebra $\mathcal{L}$ there is a coalition $A \in \mathcal{U}$ such that there exists a coalition $B \in \mathcal{L} \setminus \mathcal{U}$ with $\mu(A) = \mu(B)$.

(R1) holds whenever there is a finite partition of the set of agents into elements of the coalition algebra $\mathcal{L}$ such that for every coalition in the partition there is another coalition in the partition with the same measure. The reason is that, given any ultrafilter, one (and only one) of the coalitions in the partition should belong to the ultrafilter. In particular, (R1) would be necessarily satisfied if we replicate an arbitrary society. As long as (R1) holds, replacing non-dictatorship with μ-anonymity does result in an impossibility.

Theorem 1 *Let the Universal Domain assumption* (UD) *hold. There is no preference aggregation rule satisfying weak Paretianism* (P1), *IIA* (P2), *transitivity* (P4), *and* $\mu-$*anonymity* (P5) *if and only if assumption* (R1) *holds.*

PROOF: This proof is similar to the one Mihara [20] provides for a cardinality-based anonymity axiom. Suppose first that there is a social choice rule f satisfying (P1), (P2), (P4) and (P5). From (P1), (P2) and (P4), the set of decisive coalitions is an ultrafilter $\mathcal{U}$. (This result holds regardless of whether the society is finite or infinite; see e.g. Austen-Smith and Banks [6], p. 47).

If (R1) holds, there exist $A \in \mathcal{U}$ and $B \in \mathcal{L} \setminus \mathcal{U}$ such that $\mu(A) = \mu(B)$. Suppose now that $A \cap B = \emptyset$. Given any two alternatives x and y, consider two preference relations R and R' such that xPy and $yP'x$ and a preference profile ρ such that $\rho^{-1}(R) = A$ and $\rho^{-1}(R') = A^c$. Now define a new preference profile ρ' by $(\rho')^{-1}(R) = B$ and $(\rho')^{-1}(R') = B^c$. Since the induced distribution on preferences is the same, (P5) implies that $f(\rho) = f(\rho')$. By independence of irrelevant alternatives (P2), B must be decisive for x against y, and since x and y are arbitrary, it follows that B is decisive, contradicting our initial assumption that $B \notin \mathcal{U}$.

Suppose now $A \cap B \neq \emptyset$. Since $\mathcal{U}$ is an ultrafilter and $B \notin \mathcal{U}$, we have $B^c \in \mathcal{U}$. Therefore $A \setminus B = A \cap B^c \in \mathcal{U}$, while $B \setminus A = B \cap A^c \notin \mathcal{U}$ (otherwise $\mathcal{U}$ would contain two sets with an empty intersection, A and $B \setminus A$). We have

$$\mu(A \setminus B) = \mu(A) - \mu(A \cap B) = \mu(B) - \mu(A \cap B) = \mu(B \setminus A),$$

so the argument of the previous paragraph can be applied to $A \setminus B$ and $B \setminus A$.

Now suppose that there is an ultrafilter $\mathcal{U}$ that violates assumption (R1). Then the simple rule whose decisive coalitions are all members of $\mathcal{U}$ satisfies all the desired properties. □

It follows from this result that for any extension of the $n-$period measure space with $n \geq 2$ there is no aggregation rule satisfying (P1), (P2), (P4), and (P5). More generally, there is no aggregation rule satisfying (P1), (P2), (P4), and (P5) in every measure space in which there is a finite partition of the set of agents in equal-measure coalitions.

4.2 Consensual Rules

Transitivity of preferences may be a rather strong requirement. We next explore the possibility of relaxing it by requiring only the strict preference to be transitive.

Definition 5 A preference aggregation rule f is

(P6) *quasi-transitive* if for every $\rho \in \mathcal{R}^N_{\mathcal{L},\mathcal{E}}$ and for all $x, y, z \in X$,

$$xPy \;\&\; yPz \Rightarrow xPz.$$

For finite societies, Sen [25] showed the possibility of quasi-transitive, weakly Paretian, independent of irrelevant alternatives, and anonymous preference aggregation rules. The example proposed by Sen is the strong Pareto rule: an alternative is strictly preferred to another if and only if it is weakly preferred by everyone and strictly preferred at least by someone. There are other possibilities; from a characterization of quasi-transitive, weakly Paretian, and independent rules by Gibbard (in an unpublished paper referred to in [26]) it follows that in all possible rules that are also anonymous the only decisive coalition is the entire set of agents, and each agent has veto power. Note that Sen's example satisfies neutrality, a requirement that is stronger than independence of irrelevant alternatives and that mixes the idea of independence with that of equal-treatment of alternatives.

Let us introduce a new piece of notation. Given a preference relation R and two alternatives $x, y \in X$, let us denote by $I(R)(x, y)$ the indicator of the preferences between the ordered pair (x, y): $I(R)(x, y)$ equals 1 if xPy, 0 if xIy, and -1 if yPx. We define neutrality as follows:

Definition 6 A preference aggregation rule f is

(P7) *neutral* if, for all $\rho, \rho' \in \mathcal{R}^N_{\mathcal{L},\mathcal{E}}$ and all $x, y, a, b \in X$,

$$[\,\forall i \in N,\; I(\rho_i)(x, y) = I(\rho'_i)(a, b)\,] \;\Rightarrow\; [\,I[f(\rho)](x, y) = I[f(\rho')](a, b)\,].$$

It turns out that, under another richness condition on the coalition measure space, we can obtain results for large societies that are an extension of those that hold for finite societies. In particular, we can obtain a characterization of the class of rules which give veto power to any coalition of positive measure and, furthermore, take into account only how large is the set of agents that strictly prefer one alternative over another. Formally, we define the following class of rules:

Definition 7 Given a measure space $(N, \mathcal{L}, \mu)$, a preference aggregation rule f is *consensual* if, for every ordered pair of alternatives (x, y),

$$xPy \iff (\{i : xP_iy\}, \{i : yP_ix\}) \in \mathcal{V}_f$$

where $\mathcal{V}_f$ is a collection of ordered pairs of disjoint coalitions in $\mathcal{L}$ satisfying

(i) $(N, \emptyset) \in \mathcal{V}_f$,

(ii) If $F \cap A = \emptyset$ and $F' \cap A' = \emptyset$, then

$$[\mu(F) \geq \mu(F'),\ \mu(A) \leq \mu(A')\ \&\ (F', A') \in \mathcal{V}_f] \Rightarrow (F, A) \in \mathcal{V}_f,$$

(iii) $(F, A) \in \mathcal{V}_f \Rightarrow \mu(F) > 0 = \mu(A)$.

It is important to note that, in order for a consensual rule to be well defined, (UD) is a necessary assumption. It is a formal exercise to show that the strict preference defined as above is asymmetric and transitive, and that the corresponding weak relation (xRy if and only if [not yPx]) is reflexive and complete, so f is a well defined aggregation rule. Note, however, that the strict social preference need not be negatively transitive, *i.e.* the weak social preference need not be transitive. We also have that, since the rule only depends on the distribution of preferences, it satisfies anonymity. Finally, the rule satisfies neutrality because it is only based on the measures of agents that rank the alternatives, not on any specific characteristic of the alternatives themselves.

An extreme example of a consensual rule is almost-sure unanimity, that is, the rule defined by $(F, A) \in \mathcal{V}_f$ if and only if $F \cap A = \emptyset$, $\mu(F) = 1$ and $\mu(A) = 0$. On the opposite extreme, we have an equivalent to the strong Pareto rule defined by Sen [25]: $(F, A) \in \mathcal{V}_f$ if and only if $F \cap A = \emptyset$, $\mu(F) > 0$ and $\mu(A) = 0$.

Our richness condition on the coalition measure space is the following:

Assumption (R2): For every pair of sets B and C in $\mathcal{L}$ such that $\mu(B) > \mu(C)$, there is a set $D \subset B$ in $\mathcal{L}$ such that $\mu(D) = \mu(C)$.

(R2) is satisfied if for each finite partition of the agent space into admissible coalitions there exists a finer finite equipartition into admissible coalitions. It is obviously satisfied by any non-atomic countably additive measure over a $\sigma-$algebra. As follows from Fey [11] (Lemma 2), this property is also satisfied, for instance, by periodic measure spaces over the countable agent space. Note that neither (R1) implies (R2) nor vice versa. For instance, the cofinite measure space satisfies (R2) but not (R1). For an example that satisfies (R1) but not (R2), consider the $(2, \beta)$-discounting measure space with $\beta < 1/2$. In this case, the simple rule that has as decisive coalitions all coalitions that include both agents 3 and 4 satisfies weak Paretianism (P1), $\mu-$anonymity (P5), quasi-transitivity (P6) and neutrality (P7), even though the coalition of agents 3 and 4 is not full measure. When (R2) holds, however the following characterization is obtained.

Theorem 2 *Let assumptions* (UD) *and* (R2) *hold. Then, a preference aggregation rule satisfies weak Paretianism* (P1)*, $\mu-$anonymity* (P5)*, quasi-transitivity* (P6)*, and neutrality* (P7)*, if and only if it is a consensual rule.*

PROOF: As mentioned above, consensual rules satisfy (P1), (P5), (P6), and (P7).

For the opposite implication, suppose that there is a social choice rule f satisfying (P1), (P5), (P6) and (P7). Actually, only the weaker condition of independence rather than neutrality is needed for the first two steps of the proof. Using neutrality throughout the proof can indeed make it somewhat shorter, but we choose to present a longer version for the sake of transparency.

From (P1) we know that N is decisive. Since $\mu(N) = 1$, anonymity (P5) implies that every full measure coalition is decisive. First, we claim that there are no other decisive coalitions. Suppose this is not the case, *i.e.* there is some decisive B such that $\mu(B) < 1$. If $\mu(B) \leq 1/2$, by (R2) we can find a subset C of B^c such that $\mu(C) = \mu(B)$. But then, using anonymity (P5), we would have that C is decisive and disjoint with B, a contradiction. If $1/2 < \mu(B) < 1$, by (R2) we can find a subset C of B such that $\mu(C) = \mu(B^c)$. Then, using anonymity (P5), the coalition C^c should be decisive. But then it is easy to show that $B \cap C^c$ must be decisive. Consider any three alternatives

x, y, z and any profile ρ such that xP_iy for every $i \in B$ and yP_iz for every $i \in C^c$. Then by quasi-transitivity xPz with xP_iz for every $i \in B \cap C^c$ and with preferences for other individuals unrestricted. From independence and the fact that x and z are arbitrary, we get that $B \cap C^c$ is decisive. If the original set B had measure $\mu(B) = 1 - \varepsilon$, with $0 < \varepsilon < 1/2$, then we are left with a decisive set $B \cap C^c = B \setminus C$ that has measure $\mu(B \setminus C) = 1 - 2\varepsilon$. Iterating the process, we obtain decisive sets that have a measure of 1 minus a multiple of ε, so in a finite number of steps we get a decisive set of measure smaller than 1/2, which leads to a contradiction by the argument above.

Second, we claim that for any pair of alternatives (x, y) and any profile ρ such that $\mu(\{i : xP_iy\}) > 0$, we have xRy. That is, every positive measure coalition has veto power. For suppose that there exists some measurable profile ρ and a pair of alternatives x, y such that the coalition $A = \{i : xP_iy\}$ has positive measure, but the aggregation rule has yPx. Consider an alternative z different from x and y and a profile ρ' such that zP'_iy for every individual, and preferences over x and y are as in ρ. Then by f weakly Paretian $zP'y$ and by quasi-transitivity $zP'x$. Note that, by independence, this holds for every profile in which every agent in A^c prefers z to x, regardless of the preferences of agents in A with respect to x and z. From the usual argument behind the proof of Arrow theorem (see e.g. [6], p. 35-36), employing weak Pareto, quasi-transitivity, and independence, the coalition A^c is decisive. Since it is less than full-measure, we get a contradiction by the argument in the previous paragraph.

Now, let $\mathcal{V}_f$ be the set of pairs of coalitions (F, A) such that $F = \{i \in N : xP_iy\}$, $A = \{i \in N : yP_ix\}$, and xPy for some profile and some pair of alternatives x, y. By neutrality, if $(F, A) \in \mathcal{V}_f$, then given any pair of alternatives w and z, for any profile such that $\{i : wP_iz\} = F$ and $\{i : zP_iw\} = A$, we must have wPz. From the previous paragraphs, we know that $(N, \emptyset) \in \mathcal{V}_f$ (part (i) of the definition of consensual rules), and if $(F, A) \in \mathcal{V}_f$ then $\mu(A) = 0$. Furthermore, asymmetry of the strict preference P implies that $\mu(F) > 0$ (part (iii) of the definition). It remains to show that part (ii) of the definition holds.

Consider a disjoint pair (F, A) such that $\mu(F) \geq \mu(F')$, $\mu(A) = \mu(A') = 0$ and $(F', A') \in \mathcal{V}_f$. If $\mu(F) = \mu(F')$, by anonymity (P5) we get that $(F, A) \in \mathcal{V}_f$. If $\mu(F) > \mu(F')$, by (R2) F has a subset D of measure equal to $\mu(F')$. We have two cases. Suppose first that $\mu(F \setminus D) \leq \mu(D)$. By (R2), there exists $C \subset D$ such that $\mu(C) = \mu(F \setminus D)$. Now let $D' = F \setminus C$, then we have that $D \cup D' = F$, and $\mu(D') = \mu(F \setminus C) = \mu(F) - \mu(C) = \mu(D) = \mu(F')$.

By anonymity (P5), $(D, A) \in \mathcal{V}_f$ and $(D', A) \in \mathcal{V}_f$. Consider a profile ρ such that:

$$\begin{aligned}
i \in D \setminus D' &\Rightarrow xP_iyI_iz, \\
i \in D' \setminus D &\Rightarrow xI_iyP_iz, \\
i \in D \cap D' &\Rightarrow xP_iyP_iz, \\
i \in A &\Rightarrow zP_iyP_ix, \\
i \notin D \cup D' \cup A &\Rightarrow xI_iyI_iz.
\end{aligned}$$

We have that xPy, yPz, and, by quasi-transitivity (P6), xPz. By construction, $(F, A) \in \mathcal{V}_f$.

For the second case, if $\mu(F \setminus D) > \mu(D)$, by (R2) we can find a set $D' \subset F \setminus D$ such that $\mu(D') = \mu(D) = \mu(F')$. By (P5), $(D, A) \in \mathcal{V}_f$ and $(D', A) \in \mathcal{V}_f$. By an argument similar to the one above, $(D \cup D', A) \in \mathcal{V}_f$, and by construction $\mu(D \cup D') = 2\,\mu(F') < \mu(F)$. Taking now $(D \cup D', A)$ as starting point instead of (F', A'), we repeat the same process until, in a finite number of steps, we get a subset D'' of F such that $(D'', A) \in \mathcal{V}_f$ and $\mu(F \setminus D'') \leq \mu(D'')$, which reduces to the first case. □

In the standard finite-agent case, requiring that no single individual has veto power leads back to an impossibility result. In large societies, however, requiring that no individual has veto power is not inconsistent with consensual rules as long as every individual has (and thus, all finite coalitions have) measure zero. As discussed by Armstrong [2], assuming that individual agents have measure zero fits well with the standard notion of perfect competition.

4.3 Measuring Rules

We now drop transitivity requirements altogether, and tighten the remaining axioms by imposing monotonicity.

Definition 8 A preference aggregation rule f is

(P8) *monotonic* if for all $\rho, \rho' \in \mathcal{R}_{\mathcal{L}}^N$ and all $x, y \in X$,

$$[\, \{i : xP_iy\} \subset \{i : xP'_iy\}\,,\ \{i : xR_iy\} \subset \{i : xR'_iy\}\,,\ xPy\,] \Rightarrow xP'y.$$

For finite societies, it is known that a class of aggregation rules defined by (cardinality-based) anonymity, neutrality and monotonicity involves disregarding all information other than the head-count of members agreeing in their rankings. These are called *counting rules* (see e.g. Austen-Smith and Banks [6]). For infinite societies we can analogously define *measuring rules*, which ignore all information other than how large the coalitions preferring one alternative over another are. Formally:

Definition 9 Given a measure space $(N, \mathcal{L}, \mu)$, a preference aggregation rule f is a *measuring rule* if, for every ordered pair of alternatives (x, y),

$$xPy \iff (\{i : xP_iy\}, \{i : yP_ix\}) \in \mathcal{W}_f$$

where $\mathcal{W}_f$ is a collection of ordered pairs of disjoint coalitions in $\mathcal{L}$ satisfying

(*i*) $(N, \emptyset) \in \mathcal{W}_f$,

(*ii*) If $F \cap A = \emptyset$ and $F' \cap A' = \emptyset$, then

$$[\mu(F) \geq \mu(F'),\ \mu(A) \leq \mu(A') \ \&\ (F', A') \in \mathcal{W}_f] \Rightarrow (F, A) \in \mathcal{W}_f,$$

(*iii*) $(F, A) \in \mathcal{W}_f \Rightarrow \mu(F) > \mu(A)$.

As in the case of consensual rules, we begin by noting that (UD) is a necessary assumption for measuring rules to be well defined. Note also that consensual rules are just a special case of measuring rules with the added requirement that $(F, A) \in \mathcal{W}_f$ implies $\mu(A) = 0$. Condition (*iii*) in the definition implies that the rule is asymmetric. Reflexivity and completeness of the induced weak relation are immediate, so measuring rules are well defined aggregation rules. Measuring rules satisfy weak Pareto (P1) (because of property (*i*) in the definition), anonymity (P5) (because the rule only depends on the distribution of preferences), neutrality (P7) (because the rule only depends on the alternatives being compared insofar as they determine the measures of the sets of individuals that rank the two alternatives differently), and monotonicity (P8) (because of property (*ii*) in the definition).

It is natural to ask whether $\mu-$anonymity together with monotonicity and neutrality characterize measuring rules. The answer to this depends on the properties of the coalition measure space. When (R2) holds, the following characterization is straightforward:

Theorem 3 *Let assumptions* (UD) *and* (R2) *hold. Then, a measurable social choice rule satisfies weak Paretianism* (P1)*, μ-anonymity* (P5)*, neutrality* (P7)*, and monotonicity* (P8)*, if and only if it is a measuring rule.*

Proof: As stated above, measuring rules satisfy (P1), (P5), (P7), and (P8).

Now, suppose f is a weakly Paretian, μ-anonymous, monotonic and neutral rule. Let $\mathcal{W}_f$ be the set of pairs of coalitions (F, A) such that $F = \{i : xP_iy\}$, $A = \{i : yP_ix\}$, and xPy for some measurable profile and some pair of alternatives x, y. By neutrality, if $(F, A) \in \mathcal{W}_f$, then given any pair of alternatives w and z, for any profile such that $\{i : wP_iz\} = F$ and $\{i : zP_iw\} = A$, we must have wPz. From weak Paretianism we have that $(N, \emptyset) \in \mathcal{W}_f$. This proves part (i) of the definition of measuring rules. For part (ii), consider a disjoint pair of coalitions (F, A) such that $\mu(F) \geq \mu(F')$ and $\mu(A) \leq \mu(A')$ for some pair $(F', A') \in \mathcal{W}_f$. From (R2), there exists an admissible subcoalition C of F such that $\mu(C) = \mu(F')$. If $\mu(A) = \mu(A')$, then by anonymity we have that $(F', A') \in \mathcal{W}_f$ implies $(C, A) \in \mathcal{W}_f$. If $\mu(A) < \mu(A')$, apply (R2) to get $D' \subset A'$ such that $\mu(D') = \mu(A)$. We have $\mu(C^c \setminus A) = \mu(C^c) - \mu(A) = \mu[(F')^c] - \mu(D') \geq \mu(A') - \mu(D') = \mu(A' \setminus D')$. Apply again (R2) to get $D'' \subset C^c \setminus A$ such that $\mu(D'') = \mu(A' \setminus D')$. If we now let $D = A \cup D''$, we have $\mu(D) = \mu(A) + \mu(D'') = \mu(A')$. By anonymity, we have also have in this case that $(F', A') \in \mathcal{W}_f$ implies $(C, D) \in \mathcal{W}_f$. By monotonicity, it follows that $(F, A) \in \mathcal{W}_f$.

Part (iii) of the definition follows from the asymmetry of P and part (ii). □

It should be noted that, for the particular case in which the cardinality of the alternative space X is equal to 2 and the coalition measure space is the periodic $(\mathbb{N}, \mathcal{L}_p, \mu_p)$, this result almost exactly follows from Fey [11] (Theorems 3 and 4); the measuring rules are, essentially, the density measure $q-$rules in this author's terminology. The major difference between his and our results arises from the fact that the "bounded anonymity" of Lauwers [17], which Fey [11] requires, is only equivalent to anonymity under the periodic "density" measure for infinite coalitions with infinite complements. In a sense, anonymity with respect to the periodic measure can be viewed as the bounded anonymity with the additional requirement of negligibility of finite coalitions.

We can find examples where (R2) fails, and, nonetheless, only measuring rules satisfy (P1), (P5), (P7) and (P8). One such example is a countable society with a purely atomic measure such that $\mu(\{1\}) = \frac{1}{2}$, $\mu(\{2\}) = \mu(\{3\}) = \frac{1}{4}$ and $\mu(\{i\}) = 0$ for $i \geq 4$. Examples like this can be considered an anomaly, dependent on having no more than two basic coalition sizes that "fit" well together.

If there are atoms of three or more distinct sizes, we can always find non-measuring rules that satisfy (P1), (P5), (P7) and (P8), as can be seen from the following example, also with a countable society. Let $\mu(\{1\}) = \frac{1}{2}, \mu(\{2\}) = \frac{1}{4}, \mu(\{3\}) = \mu(\{4\}) = \frac{1}{8}$ and $\mu(\{i\}) = 0$ for $i \geq 5$. Now consider the rule where xPy if and only if xP_1y and at least one of these is satisfied: (i) xP_2y, (ii) xR_3y and xP_4y, or (iii) xP_3y and xR_4y. This rule satisfies the desirable properties, but it is not a measuring rule, because the profile xP_1y, yP_2x, xP_3y and xI_4y leads to xPy, but the profile xP'_1y, yI'_2x, xP'_3y and yP'_4x leads to $xI'y$. Note that in this example, as in the previous one, only consensual rules satisfy (P1), (P5), (P6), and (P7). However, a small perturbation of the agents' weights would lead to non-consensual rules satisfying our properties.

5 Conclusions and Future Research

This paper considers the problem of social choice in a coalition measure space. Rather than following the usual definition of anonymity as equal treatment of coalitions of the same cardinality, we define anonymity (with respect to the underlying measure) as the property that the aggregation rule depends only on the distribution induced on preferences, and this implies equal treatment of coalitions of the same measure size. Our analysis here subsumes the finite case, and we have identified conditions under which the standard results in the finite case can be extended to this more general setting. Our conditions require roughly that the measure defines sufficiently large classes of equal-size coalitions. One might pursue the task initiated here and extend our results to parallel all of the results of the finite case (see, *e.g.* [6]), as the characterization of q-rules or considering the consequences of requiring acyclic aggregation rules. Our intention here is rather to convince social scientists that the generalization formulated has the benefits of including interesting applications previously excluded from the standard models, and the corresponding costs reduce basically to introducing the language of algebras and measures.

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