Equilibria in a Dynamic Global Game: The Role of Cohort Effects*

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Abstract

We introduce strategic waiting in a global game setting with irreversible investment. Players can wait in order to make a better informed decision. We allow for cohort effects and discuss when they arise endogenously in technology adoption problems with positive contemporaneous network effects. Formally, cohort effects lead to intra-period network effects being greater than inter-period network effects. Depending on the nature of the cohort effects, the dynamic game may or may not satisfy dynamic increasing differences. If it does, our model has a unique rationalizable outcome. Otherwise, there exists parameter values for which multiple equilibria arise because players have a strong incentive to invest at the same point in time others do.

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1 Introduction

In many economic situations, the optimal action of an economic agent is complementary to the actions undertaken by other agents. For example, a consumer's payoff from buying a computer software is typically increasing in the number of other consumers who also use that software. Or, think of a consumer who decides to buy a durable consumption good such as a car. As more consumers buy this brand of car, more repair shops will have the know-how and spare parts to repair the car quickly.[1] Models of situations in which the agents' optimal actions are complementary to each other are often plagued by multiple equilibria with self-fullfilling beliefs: If a player expects the other players to buy the software, then it is in her best interest to buy it as well. If a player expects the other players not to acquire the software, she wants to refrain from buying. This multiplicity result is annoying from an economic-policy point of view. Without an adequate theory of equilibrium selection, one cannot use these theories to predict the market outcome. How then does one judge, for example, whether policies to subsidize/tax the adoption of information technology should be implemented? How does one predict the market power of firms who sell their products in markets with network externalities?

For two-player coordination games, Carlsson and van Damme (1993), henceforth CvD, developed an equilibrium selection theory, which was adopted to a coordination problem with a continuum of players by Morris and Shin (1998). CvD assume that the agents' payoffs depend on the action chosen by the other agent in the economy and some unknown economic fundamental summarized by the state of the world θ. Agents receive different signals about θ, which generate beliefs about the state of the world and a hierarchy of higher order beliefs (beliefs about the other agents' beliefs, beliefs about the other agents' beliefs about her beliefs, etc.). CvD called this incomplete information game a global game and showed that if the potential type space is rich enough, the game has a unique equilibrium.[2]

[1] Complementarity of optimal actions is also a key ingredient of many models of macroeconomic coordination failures such as currency crises, debt crises, bank runs, financial crashes, and Keynes-type underemployment (Obstfeld (1996), Cole and Kehoe (2000), Diamond and Dybvig (1983), Bryant (1983)). Milgrom and Roberts (1990) discuss other examples of games with strategic complementarities such as R&D competition, oligopoly, coordination in teams, arms races, and pretrial bargaining.

[2] We refer to any binary action game that is characterized by strategic complementarity, incomplete information, and in which for some types it is a dominant strategy to adopt one action while for others it is a dominant strategy to adopt the other as a global game. That

Thus, the global game framework enables researchers to base policy recommendations on a theory that predicts behavior in coordination games. It has been applied to a wide variety of contexts within a static framework.[3] In reality, however, many economic coordination problems are essentially dynamic. Players can always postpone their investment decisions in order to make a better informed decision at a later point in time. In this paper, we investigate conditions under which the global game approach can be extended to model dynamic technology adoption problems.

To address this question, we build a dynamic global game. We consider a continuum of investors, who have the opportunity to engage in a risky investment project in either of two periods. Investments are irreversible. Payoffs depend positively on the realization of a random variable, which we refer to as the fundamental, and on the mass of investors. All players receive some noisy private information concerning the realization of the fundamental. For very high signals, it is a dominant strategy to invest immediately and for very low signals it is a dominant strategy not to invest. For intermediate signals, a player's optimal behavior depends on her beliefs about how other investors act. If a player decides to wait, she gets a more informative signal concerning the realization of the fundamental at the cost of foregone profits. Our signaling technology is simple in the sense that we assume a uniform distribution of period-one and period-two signals. We work with a flexible dynamic payoff structure in which a player's gain of investing not only depends on the total mass of players who invest, but also on *when* the other players invest. We say that our payoffs exhibit an early (late) mover cohort effect if the early (late) adopters enjoy more network benefits from the other early (late) adopters than from the late (early) ones. Four main results emerge from our analysis.

First, we show that cohort effects can arise endogenously in a dynamic set-up with contemporaneous network effects. We discuss three archetypical technology adoption problems. In the first, which we call "Adopting a Technology with a Fixed

heterogeneity of agents can lead to a unique equilibrium in situations in which the agents' actions are complementary to each other was shown by Postlewaite and Vives (1987) in a bank-run model. For a comprehensive survey of the global game literature, see Morris and Shin (2003). For an extension of the global game approach to many action games, see Frankel, Morris and Pauzner (2003).

[3]It is used, for example, to model currency crises (Morris and Shin (1998), Corsetti et al. (2004)), bank runs (Goldstein and Pauzner (2003), Rochet and Vives (2002)) and car-dealer markets (Dönges and Heinemann (2000)).

Lifespan" (FL), players decide to adopt a technology that becomes obsolete after two periods. In the interim period in which late movers have not invested yet, early movers are subject to a contemporaneous network effect that only depends on the mass of early movers. Since the technology of the early movers becomes obsolete while late movers are still using the technology, late movers benefit from a contemporaneous network effect in period 3 that depends only on the mass of late movers. This interpretation is thus characterized by an early and a late mover cohort effect. In the second interpretation, which we call "Joining a Nascent Club" (NC), players must decide whether to become member of a club. The more players who join the club, the greater is its attractiveness. At time one the club's member base may grow in the future. At time two the club's member base has reached a "mature" level (and will remain constant in the future). For the same reason as the one explained above, this interpretation exhibits an early mover cohort effect. For late members, however, the network benefit in any period depends on the total mass of members (independently on when the other players joined the club). Hence, this interpretation is void of any late mover cohort effect. In the last interpretation, which we refer to as "Pledging to Invest" (PI), players commit whether or not to invest before the technology becomes available. As the technology becomes available to all players at the same point in time, there is neither an early nor a late mover cohort effect.

We next introduce a condition on the ex post payoff function called *dynamic increasing differences*. Call both a change from not investing at all to investing at time two and a change from investing at time two to investing at time one, a move to a higher action.[4] Dynamic increasing differences implies that as a higher percentage of the population takes a higher action, it becomes weakly more profitable to take a higher action. For example, it requires that as more players invest late rather than not at all, it becomes weakly more profitable to invest at time one. Our second result shows that a technology adoption problem that exhibits contemporaneous increasing differences does not necessarily exhibit dynamic increasing differences. In particular, dynamic increasing differences requires there to be no late mover cohort effect and it is thus violated by the FL interpretation discussed above. The other interpretations, however, satisfy dynamic increasing differences.

[4]This definition is based on an ex-post perspective of the "action space" according to which a player either invested in period one, invested in period two, or did not invest at all. Clearly, the definition is not based on the normal form action space (i.e. the set of pure strategies) nor on the extensive form game as we, for simplicity, ignore the (infinite) set of histories. A rigorous definition of dynamic increasing differences is provided below.

Our third main result proves that dynamic increasing differences indeed implies the existence of a unique rationalizable outcome. We start by observing that active players who have a "very high" second-period signal always want to invest,[5] since they believe that the fundamental is so good that investing is profitable no matter what actions the other players choose. Now consider a player who has an "extremely high" first-period signal so that she foresees that her second-period signal will be very high even if she gets bad news. Obviously, she strictly benefits from investing immediately and saving the waiting costs if she expect no other player to invest.

Now consider a player who has a "high" but not an "extremely high" first-period signal. If she expects no other player to invest in either period, then she would also prefer to refrain from investing. As she possesses a flat prior concerning the realization of the fundamental, it is equally likely that the other players received a higher or lower signal than hers. Therefore, in equilibrium, she cannot expect no other player to invest. As her signal is "high," her knowledge that all players with an extremely high signal invest at time one and that all active players with a very high signal invest at time two induces her to invest at time one as well. Similarly, the knowledge that all players with an extremely high signal invest at time one and that all active players with a very high signal invest at time two, gives active players with a high (but not a very high) signal a strict incentive to invest at time two. This will, in turn, convince players with slightly less favorable signals to also invest, etc... . This process of iterative elimination of dominated strategies ends at some cutoff vector $(\overline{k}_1, \overline{k}_2)$. At this cutoff vector a player with a first-period signal equal to $\overline{k}_1$ is indifferent between investing at time one and waiting if she expects all active players to invest in period t whenever they receive a signal above $\overline{k}_t$ and refrain from investing otherwise. Similarly, a player with a second-period signal equal to $\overline{k}_2$ who has the above expectation about the other players' behaviors is indifferent between investing and not investing. Mirroring the above argument, because it is a dominant strategy not to invest for very low signals, players with low signals refrain from investing. After iteratively deleting strategies from below, there is a critical cutoff vector $(\underline{k}_1, \underline{k}_2)$ such that a player refrains from investing in period t whenever she has a signal below $\underline{k}_t$. We next observe that these cutoff vectors give rise to symmetric switching equilibria. In the final step, we show that if the ex post payoff function satisfies dynamic increasing

[5] A player is said to be active if she has not invested yet.

differences then the extremal switching equilibria coincide, i.e. $(\overline{k}_1, \overline{k}_2) = (\underline{k}_1, \underline{k}_2)$. In contrast to the earlier steps of the proof, which extend arguments from the static global game literature to our dynamic set-up, the final step is based on a completely novel argument exploiting the nature of symmetric switching equilibria.

Fourth, we characterize symmetric switching equilibria for a wide range of parameter values. We find closed form solutions for all economic interpretations introduced earlier. For the economic interpretations that satisfy dynamic increasing differences, we thus characterized the unique symmetric switching equilibrium. This permits us to analyze how changes in the investment cost, in the discount factor, and in the precision of the players' signals affect both the timing of investments and the total investment activity. We show, for example, that as players become more patient, both late *and* early investment activity can increase in the NC interpretation. The characterization also allows us to illustrate why multiple equilibria can arise if dynamic increasing differences are violated. In essence, if dynamic increasing differences are violated, then players have an incentive to invest at the same point in time at which other players do. If this incentive is strong enough, it gives rise to self-fullfilling expectations according to which some players invest at time two if and only if they anticipate other players to do the same. We also give a necessary and sufficient condition for the fixed lifespan interpretation to have a unique equilibrium within the class of symmetric switching equilibria. Thus, at least within the class of symmetric switching equilibria, dynamic increasing differences are not necessary for the uniqueness of equilibrium.

This is not the only paper to introduce dynamic elements in a global game. Chamley (1999) studies a dynamic global game in which there is some uncertainty about the distribution of the investment costs. The distribution of the investment costs evolves stochastically through time. Players use the observed previous aggregate behavior together with their knowledge of equilibrium strategies, to update their beliefs about the state of the world. As long as there is sufficient heterogeneity in the population, each period can be analyzed as a static global game and, hence, there is a unique equilibrium.[6] A key difference to our paper is that there is a new

[6]In a static global game set-up, if players observe a public and a private signal and the public signal is sufficiently precise, then multiple equilibria prevail as players can use the public signal as a coordination device (see Hellwig (2002)). In Chamley's model, if the population is sufficiently heterogenous, the inferences players draw based on past observed behavior are less precise. Because the parameter of the distribution of investment costs changes only slowly over time, heterogeneity is needed to rule out equilibria in which players ignore their private signals

population of players in every period. Thus, players cannot choose when to invest.

Morris and Shin (1999) study the onset of currency crises using a dynamic global game in which the fundamental follows a Markov process. As long as their has been no successful attack, all players chooses whether or not to attack in every period. In each period the past realizations of the fundamental is common knowledge and players observe a private signal regarding the current realization. If the private signal is sufficiently precise, each period can be analyzed as a static global game and the model has a unique equilibrium.[7] In contrast to our model, investments are not irreversible.

Burdzy, Frankel, and Pauzner (2001) investigate a dynamic model in which the state evolves stochastically through time and at each point in time there is common knowledge about the current state and past choices. In each period a continuum of players is randomly matched to play a 2x2 game with strategic complementarities. Under the assumptions that (i) in some states of the world playing one action is dominant while in others the other is dominant and (ii) that in each period a player has only a small chance of revising her action, they show that players choose to play the risk-dominant equilibrium in the limit as revision opportunities arrive quickly.[8] Frankel and Pauzner (2000) use a similar set-up to investigate a model of sectoral choice in which there are external increasing returns and show that there is a unique equilibrium even if frictions are nonnegligible. Oyama (2003) also uses a similar set-up to analyse economic fluctuations in less developed countries. In none of those papers, players can engage in strategic waiting. Also, the argument in those papers relies on only a small set of players being able to revise their action at any given point in time, while in our paper all players can move at the same time.

Dasgupta (2001) introduces elements of strategic waiting in a global game with irreversible investment. Players can invest in two periods. If a player delays, she

and use their common *equilibrium knowledge* of the distribution of the states of the world as a "public signal" to coordinate behavior.

[7]Toxvaerd (2002) analyzes merger waves using a similar set-up as Morris and Shin (1999). Maintaining the assumption that all past realizations of the fundamental are common knowledge, Toxvaerd and Giannitsarou (2003) provide a general framework for finite horizon models in which each period can be analyzed as a static global game.

[8]Levin (2001) adopts this framework to considers a many action game in which players move according to an exogenous sequence. He shows that the equilibrium is unique under a no influence condition, while otherwise there may be multiple equilibria.

observes a noisy signal about the past economic activity at the cost of foregone profits. Dasgupta shows that his game, under some additional assumptions on the prior distribution and the signaling technology, is characterized by a unique equilibrium within the class of switching strategies. The main difference between our paper and his is that we investigate cohort effects, which are not present in his model in which payoffs depend only on whether a sufficient number of players invest in either period. Another difference is that in his model one wants to delay to engage in social learning, while in our model a player delays to obtain a more precise signal. Furthermore, when establishing uniqueness in the absence of cohort effects, we do not restrict attention to switching strategies only.

Echenique (2004) investigates the set of subgame perfect equilibria in extensive-form dynamic games with strategic complementarities using a lattice theoretical approach. Technically, as our game has no proper subgame, his results are of limited use in our set-up. Furthermore, Echenique does not address the central question of our paper: Under what conditions is equilibrium unique? Economically, however, he also observes that static strategic complementarities do not imply dynamic complementarities though for different reason than in our set-up. He defines a game as having dynamic strategic complementarity, whenever a switch to a higher action following any given history, implies that the rivals choose a (weakly) higher actions following every history. Echenique observes that extremal outcomes (e.g. cooperation in the infinitely repeated prisoners' dilemma) may rely on "punishments" or taking low actions following certain histories. If a player switches to taking high actions following such punishment histories also (e.g. always cooperates in the prisoners' dilemma independent of past play), then the best reply of his opponent is to sometimes take a lower action (i.e. defect rather than cooperate). Thus, even if the static game is one of strategic complementarity, the dynamic game may not be. Note that this argument is based on the observation of past play, which we abstract from. Rather, in our paper multiple equilibria are driven by cohort effects, which are endogenously derived from primitives of different technology adoption problems.

The remainder of this paper is organized as follows. In section 2, we introduce our formal model. In section 3, we relate the parameters of our model to different economic environments. In section 4, we analyze equilibrium behavior in our model. We first state our definition of dynamic increasing differences and show that, when our ex post payoff function satisfies that condition, our model features an essentially unique outcome (subsection 4.1). Next, in subsection 4.2, we provide closed-form solutions for two important types of symmetric switching equilibria and we analyze when (i.e. for which values of our exogenous parameters) our model has multiple equilibria. In section 4.3 we detail on the basis of two (pedagogical) examples the driving force behind our multiplicity result. Section 5 presents some comparative static results when our model features an essentially unique equilibrium. Final comments are summarized in section 6. All proofs can be found in the appendix.

2 The model

Assume a continuum of risk-neutral players with mass one. All players have the opportunity to undertake one risky investment project. Investments are irreversible. A player can invest at time one, at time two, or can decide not to invest at all. If player i decides to invest at time one, she gets a utility U_1^i equal to:

$$U_1^i = \theta + n_1 + \alpha n_2 - 1,$$

where n_1 (n_2) denotes the mass of players who invest at time one (two). The state of the world θ is randomly drawn from a uniform distribution along the entire real line. A period-two investor enjoys a utility equal to:

$$U_2^i = \tau(\theta + \gamma n_1 + n_2 - 1 - \Delta).$$

If player i decides not to invest in both periods, she gets zero. Throughout, we assume that $\tau, \alpha, \gamma \in [0, 1]$ and that $\Delta \geq 0$. We postpone the discussion of the economic motivation for our payoff structure until the next section.

All players possess a private and imperfect signal concerning the realized state of the world. Formally, player i's first-period signal, s_1^i, equals:

$$s_1^i = \theta + \epsilon_2^i + \epsilon_1^i,$$

and his second period signal, s_2^i, equals:

$$s_2^i = \theta + \epsilon_2^i.$$

The errors ϵ_2^i are uniformly distributed in the population over the interval $[-\epsilon, \epsilon]$. Half of the population receives an error $\epsilon_1^i = -\epsilon$, and half of the population receives an error $\epsilon_1^i = \epsilon$. Errors ϵ_1^i and ϵ_2^i are independently distributed in the population.

Note that our model possesses some "desirable" features that highly simplify the computation of our equilibrium strategies and enable a direct comparison with the static counterparts of our model. First, note that s_1^i is constructed by adding noise to s_2^i. In statistical terms, this means that s_2^i is a sufficient statistic for s_1^i. In particular, this implies that $E(\theta|s_2^i, s_1^i) = E(\theta|s_2^i)$. Second, we know that $\theta = s_2^i - \epsilon_2^i$. Hence, $\theta|s_2^i \sim U[s_2^i - \epsilon, s_2^i + \epsilon]$, and $E(\theta|s_2^i) = s_2^i$. Third, the first-period signals are also uniformly distributed around θ. To see this, consider the following figure:

[Insert here Figure One]

The lower part of Figure One represents the time-two distribution of signals. The upper part of Figure One represents the time-one distribution of signals. From above, we know that at time two signals are uniformly distributed between $\theta - \epsilon$ and $\theta + \epsilon$. Suppose $s_2^i \in [\theta, \theta + \epsilon]$. If player i's first period noise equals $-\epsilon$, $s_1^i \in [\theta - \epsilon, \theta]$. If player i's first period noise equals $+\epsilon$, then her first period signal $s_1^i \in [\theta + \epsilon, \theta + 2\epsilon]$. The same argument also applies to a player whose second-period signal lies between $\theta - \epsilon$ and θ: Depending on the realization of her first-period noise, she will either lie between $\theta - 2\epsilon$ and $\theta - \epsilon$ or between θ and $\theta + \epsilon$. As the prior probability that $\epsilon_1^i = \epsilon$ equals $\frac{1}{2}$, it follows that $s_1^i \sim U[\theta - 2\epsilon, \theta + 2\epsilon]$. Fourth, one can apply a similar logic to show that $\theta|s_1^i \sim U[s_1^i - 2\epsilon, s_1^i + 2\epsilon]$. Finally, note that $E(\theta|s_1^i) = s_1^i$. As our errors are uniformly distributed in both periods, it follows from Morris and Shin (2003) that if players were allowed to invest either only in the first or only in the second period, our game would be characterized by an essentially unique equilibrium in rationalizable strategies.

The timing of the game is as follows:
0) Nature chooses θ. All players receive their first-period signals.
1) All players simultaneously decide whether to invest or wait.
2) Player i observes whether $\epsilon_1^i = \epsilon$ or $\epsilon_1^i = -\epsilon$ but not n_1. If she did not invest at time one, she decides whether or not to do so at time two.
3) All players receive their payoffs and the game ends.

Each player's time-one action space, A^1, equals {invest, not invest}. Player i's time-two action space, A^2, equals {invest, not invest} if $a_1^i =$ not invest, and equals {not invest} if $a_1^i =$ invest. Player i's observable history at time one is $H_1^i = \{s_1^i | s_1^i \in \Re\}$

and at time two is $H_2^i = \{(s_1^i, s_2^i) | s_1^i \in \Re, s_2^i \in \{s_1^i - \epsilon, s_1^i + \epsilon\}\} \times A^1$. Let $\sigma^i = (\sigma_1^i, \sigma_2^i)$ denote player i's behavioral strategy, where $\sigma_1^i(s_1^i)$ represents the probability with which player i invests at time one given her first period signal and $\sigma_2^i(s_1^i, s_2^i)$ represents the probability with which player i invests at time two given (s_1^i, s_2^i) and given that she did not invest in the first period. (Trivially, a player cannot invest in the second period if a_1^i =invest, i.e. if she already invested in the first period.) We denote a strategy profile by σ.

Frequently, we will refer to symmetric switching strategies. A strategy profile is a symmetric switching strategy profile if it can be parameterized by a single vector $k \equiv (k_1, k_2)$ with the interpretation that: (i) $\sigma^i(s_1^i) = 1$ if and only if $s_1^i > k_1$, (ii) $\sigma^i(s_1^i, s_2^i) = 1$ if and only if $s_2^i > k_2$ for all i. An equilibrium in symmetric switching strategies is a k^* such that player i's strategy is a best response at every information set given (i) her beliefs about the state of the world, and given (ii) the equilibrium behavior of all other agents.

3 Economic Interpretations

The general payoff structure of our model nests a wide variety of more specific models. We provide three detailed interpretations below.

1 Adopting a Technology with a Fixed Lifespan (FL). This interpretation is most appropriate in an environment characterized by technological progress (e.g. consumer electronics, computers, etc...). Suppose players can invest in a new technology with an unknown quality. This technology exhibits positive network effects and can be used for T periods. For simplicity, players are only allowed to invest in period 1 or period 2 and have a common discount factor δ. Call a player who invests at time one (two) an (a) early (late) adopter. When investing, players need to pay a setup cost $s \geq 0$. The (net of any per-period cost) return of the investment in period t $(t = 1, ..., T)$, is given by $v_t^i = \tilde{\theta} + m_t$, where m_t denotes the mass of players who are using the technology at time t. Assume, for the sake of simplicity, that $T = 2$. In this case the payoff of a player investing in period 1 is given by

$$V_1^i = (1+\delta)\tilde{\theta} + (1+\delta)n_1 + \delta n_2 - s,$$

and of a player investing in period 2 is given by

$$V_2^i = \delta(1+\delta)\tilde{\theta} + \delta n_1 + \delta(1+\delta)n_2 - \delta s.$$

Setting $\theta = \tilde{\theta} - \frac{s}{(1+\delta)} + 1$ and using the utility transformation $U_t^i = \frac{V_t^i}{(1+\delta)}$ shows that this economic model is a special case of our model in which $\tau = \delta$, $\alpha = \delta\gamma < \gamma = \frac{1}{1+\delta} < 1$, and $\Delta = 0$.

To illustrate this interpretation, suppose everyone has the opportunity to buy a video player. The more people who buy a video player, the higher the availability of video movies, video rental stores, etc... . A video player can only be used for two periods. Everyone knows that at time three the DVD player will be introduced in our economy. As DVD technology is superior to video technology, from time three on, no one wants to buy a new video player anymore. However, people only switch to the superior DVD technology once their video player becomes "too old" (i.e. early adopters switch to the superior technology at time three, while late adopters switch to the new technology at time four). Whenever $\alpha < 1$ ($\gamma < 1$), we say that our model exhibits an early (late) mover *cohort effect*. Hence, our FL interpretation exhibits early and late mover cohort effects. This is intuitive: At time one the early movers do not enjoy any network benefits from the late movers. Therefore early movers care more about the mass of players who adopt the technology at time one than about the mass of players who adopt it at time two (which explains why in this case $\alpha < 1$). Late movers know that the installed base will become smaller at time three due to the early movers' switching to the new technology. Therefore, $\gamma < 1$.

2 Joining a Nascent Club (NC). In this interpretation all players must decide whether or not to join a "secret" club like Freemasonry, the Rosicrucian movement, Opus Dei, etc. . Call period one the "start-up" phase. Period two represents the "mature" phase. The larger the club's member base, the greater the utility it provides to all its members. In period one, the movement is still relatively new and its member base may grow in the future. In period two, the club's member base has reached its mature level and remains constant thereafter. To become a member of the club, players must bear a fixed cost equal to $F \geq 0$.[9] Once someone becomes a member, she must pay a per-period membership fee (denoted by $s \geq 0$). If a player joins the club at time one, she gets

$$V_1^i = (\tilde{\theta} + n_1 - F - s) + \delta(\tilde{\theta} + n_1 + n_2 - s) + \delta^2(\tilde{\theta} + n_1 + n_2 - s) + ...,$$

[9] For example, people can only join the Rosicrucian movement after having shown to possess enough knowledge of the Bible. Similarly, to become a Freemason, one must also pass a series of tests.

$$= \frac{1}{1-\delta}(\tilde{\theta} + n_1 + \delta n_2 - s) - F.$$

If she joins the club at time two, she gets

$$V_2^i = \frac{\delta}{1-\delta}(\tilde{\theta} + n_1 + n_2 - s) - \delta F.$$

Setting $\tilde{\theta} \equiv \theta + s - 1 + (1-\delta)F$ and after applying the utility transformation $U_t^i = V_t^i(1-\delta)$, the reader can check that this interpretation is a special case of our model in which $\alpha = \tau = \delta < 1$, $\gamma = 1$ and $\Delta = 0$.[10] For the same reason as above, this interpretation possesses an early mover cohort effect. However, in this interpretation late members know that they will never suffer from the early members' switching to another club. Therefore, this interpretation is void of any late mover cohort effect.

3 Pledging to Invest (PI). Suppose there are two periods in which players can commit to invest in a future project. For example, firms may commit to buy some land in a soon-to-be developed industrial zone (or individuals may commit to become a member of some club or join a lobbying organization). In the first period, the land is sold at a lower price than in the second period (or there is a reduced membership rate). The more players who invest in either period, the better the infrastructure provided (or the more exciting it will be to visit the club or the more influential will the lobbying organization be). In period 3, all players that committed to invest pay the amount due and start getting the benefit from the planned activity. This can be captured by a model in which $\alpha = \gamma = \tau = 1$ and $\Delta > 0$. In this interpretation an early investor (or member) knows that the network externality only depends on the future size of the network (which was not the case in our FL interpretation). Hence, this interpretation is void of any cohort effects.

While the PI interpretation may seem somewhat special, it is in line with the classical papers on network externalities (e.g. Farrell and Saloner (1985)), which assume that the network effect only depends on the total number of adopters (independently of when each player adopts the technology). The preceding analysis questions the validity of this modeling choice for many environments. In particular, it seems to us that for many technology adoption problems the FL interpretation

[10] Note also that the FL interpretation coincides with the NC one if the number of periods T is infinite and per period profits are constant. In this interpretation, all players decide whether or not to construct a new plant in period one or two. Once build, each plant generates an infinite stream of constant per-period net profits (say, net of a fixed per-period maintenance cost s).

is more natural. The importance of this modeling choice is shown below.

Our payoff structure also encompasses many other interpretations. For example, suppose consumers must decide whether or not to buy a software program.[11] At time two the producer releases a new version (say, version 2.0) that is partially incompatible with the one sold at time one (say, version 1.0). In this interpretation (early and late mover) cohort effects are driven by the fact that the early and late adopters use different and only partially compatible technologies.

One special feature of our model is that it is void of any social learning, i.e. players do not observe past investment decisions. While this assumption may seem restrictive, it allows us to identify cohort effects as the driving force behind our multiplicity result. Moreover, this assumption is sometimes realistic. For example the "secret" organizations we list in our NC interpretation do not divulge the size of their member base. Similarly, imagine our players must decide whether to invest in a foreign developing country. If FDI statistics are not released, completely unreliable, or only released with a big delay, investors cannot infer how many other firms have invested (recently) and it may be natural to abstract from social learning. Similarly, the seller of a product that exhibits positive network externalities always has an incentive to exaggerate her customer base because this increases consumers' willingness to pay. She thus may not be able to credibly announce her customer base. Finally, despite this absence of social learning, our set-up is already rich enough to generate many insights. We briefly discuss the robustness of those insights with respect to the introduction of social learning below.

4 Analysis of Strategic Waiting

In this section we analyze the dynamic investment game. We first introduce some concepts that will be used throughout the following subsections. In the first subsection, we define dynamic increasing differences and show that it implies the existence of a unique rationalizable outcome of the dynamic investment game. We also show, however, that contemporaneous increasing differences does not necessarily imply dynamic increasing differences. The next subsection characterizes a class of symmetric switching equilibria. This allows us to do comparative statics when our waiting game has a unique equilibrium. We also use this characteriza-

[11] We are grateful to Larry Karp for suggesting this interpretation.

tion in the final subsection to illustrate multiplicity of equilibria in the absence of dynamic increasing differences. In essence, in the absence of dynamic increasing differences players have a strong incentive to invest when other players invest. This gives rise to a coordination problem of when to invests that allows for self-fullfilling expectations.

Let

$$h(s_2^i, \sigma) \equiv s_2^i + E(\gamma n_1 + n_2 | s_2^i, \sigma) - 1 - \Delta. \tag{1}$$

$h(s_2^i, \sigma)$ is the expected payoff of a player who invests in the second period after getting signal s_2^i, given the strategy profile σ. Similarly, we define

$$W(s_1^i, \sigma) \equiv \frac{\tau}{2} \max\{0, h(s_1^i + \epsilon, \sigma)\} + \frac{\tau}{2} \max\{0, h(s_1^i - \epsilon, \sigma)\}. \tag{2}$$

$W(s_1^i, \sigma)$ denotes the gain of waiting for player i, given her first-period signal s_1^i and given that all other players play according to σ. If player i postpones her investment decision, then with probability 1/2 she will get "bad news," i.e. she will learn that at time one she was too optimistic because $\epsilon_1^i = +\epsilon$. With probability 1/2, however, she will receive "good news" in the sense that she will learn that $\epsilon_1^i = -\epsilon$. Equation (2) states that player i's gain of waiting equals her expected second-period payoff given that she will make an optimal second-period investment decision (i.e. not invest at time two if and only if her gain of investing is negative). For brevity, define

$$g(s_1^i, \sigma) \equiv s_1^i + E(n_1 + \alpha n_2 \mid s_1^i, \sigma) - 1 - W(s_1^i, \sigma). \tag{3}$$

Trivially, it is optimal to invest in the first period for a player with a signal s_1^i (who believes that all his rivals play according to σ) if and only if $g(s_1^i, \sigma) \geq 0$.

4.1 Dynamic Increasing Differences and Uniqueness

In this subsection, we first define dynamic increasing differences, which is a condition on the ex-post payoff function. From an ex-post perspective, a player either did not invest in either period (which we refer to as action 0), invested in the second period (which we refer to as action a_2), or invested in the first period (which we refer to as action a_1). Think of not investing as the lowest action and investing in the first period as the highest action. Denote the difference in ex-post payoffs between investing in the second period and not investing by

$$\Delta U^i(a_2, 0) \equiv \tau(\theta + \gamma n_1 + n_2 - 1 - \Delta),$$

and denote the difference between investing in the first period and investing in the second period by

$$\Delta U^i(a_1, a_2) \equiv \theta + n_1 + \alpha n_2 - 1 - \tau(\theta + \gamma n_1 + n_2 - 1 - \Delta).$$

We say that the ex post payoff function exhibits *dynamic increasing differences* if and only if:

$$(i) \quad \frac{\partial \Delta U^i(a_2, 0)}{\partial n_2} = \tau \geq 0,$$

$$(ii) \quad \frac{\partial \Delta U^i(a_1, a_2)}{\partial n_2} = \alpha - \tau \geq 0,$$

$$(iii) \quad \frac{\partial \Delta U^i(a_2, 0)}{\partial n_1} - \frac{\partial \Delta U^i(a_2, 0)}{\partial n_2} = \tau(\gamma - 1) \geq 0,$$

$$(iv) \quad \frac{\partial \Delta U^i(a_1, a_2)}{\partial n_1} - \frac{\partial \Delta U^i(a_1, a_2)}{\partial n_2} = 1 + \tau - \tau\gamma - \alpha \geq 0.$$

Condition (i) states that as more players invest in the second period, investing in the second period becomes more attractive relative to not investing. This condition is implied by the fact that the contemporaneous payoff function exhibits increasing differences. Condition (ii) requires that if more players invest in period 2, it becomes weakly more profitable to invest early. Intuitively, it implies that as more players invest in the second period, there is no additional gain from switching and investing late rather than early. It thus requires that the inter-period network effect α, which measures the increase in payoff for a player who invests immediately, is no less than the discount factor τ, which measures the increase in payoff for a player who invests late. Condition (iii) states that as more players move from investing late to investing early, it becomes weakly more profitable to invest late rather than not to invest at all. Observe that this condition can only be satisfied in the absence of late mover cohort effects, i.e. if $\gamma = 1$. Finally, condition (iv) states that as more investors invest early rather than late, investing early becomes more profitable. This condition is always satisfied. We conclude that our ex post payoff function exhibits dynamic increasing differences if and only if $\alpha \geq \tau$ and $\gamma = 1$. This implies the following observation:

PROPOSITION 1 *The "Joining a Nascent Club" (NC) and the "Pledging to Invest" (PI) interpretations exhibit contemporaneous and dynamic increasing differences. The "Adopting a Technology with a Fixed Lifespan" (FL) interpretation exhibits contemporaneous but not dynamic increasing differences.*

The important consequences of whether the ex post payoff function satisfies dynamic increasing differences or not, are discussed below. We start by observing that dynamic increasing differences implies the existence of a unique rationalizable outcome in our set-up.[12]

PROPOSITION 2 *If there are positive waiting costs ($\tau < 1$ or $\Delta > 0$) and if the ex post payoff function satisfies dynamic increasing differences, there exists an essentially unique rationalizable outcome.*

Intuitively, the argument proceeds as follows. Suppose player i did not invest at time one. Then, not investing (at time two) is dominated for her whenever $s_2^i > 1 + \Delta$. Now consider any type who has a signal $s_1^i > 1 + \Delta + \epsilon$. This player foresees that she would want to invest in the second period independent of whether she receives good or bad news. But under the assumption that no other player invests, it is obvious that waiting and investing for sure is dominated by investing immediately and saving the waiting cost. Indeed, understanding that she will invest in the second period if $s_2^i > 1 + \Delta$, a player at time one can calculate her benefit of investing under the assumption that no other player invests in either period, and determine a cutoff level $\overline{s}_1^1$ such that for all higher first-period signals, she prefers to invest immediately. Dynamic increasing differences ensures that investing dominates not investing for all players who receive a signal $s_1^i > \overline{s}_1^1$ or $s_2^i > 1 + \Delta$. The reason is that as other players invest (in either of the two periods), investing early becomes even more attractive relatively to investing late, which in turn becomes even more attractive relatively to not investing at all. Call $\overline{s}^1 = (\overline{s}_1^1, 1 + \Delta)$.

A rational player anticipates that all other players invest (in one of the two periods) if they receive a sufficiently high signal. Now, consider a player with a second-period signal slightly below $1 + \Delta$ and suppose she has not invested in period one. Because a player with signal $1 + \Delta$ expects *at least* half of the population to invest, a player with a signal close to $1 + \Delta$ strictly prefers to invest. Similarly, a player with a signal $s_1^i = \overline{s}_1^1$ must expect half of the population to invest at time one and perhaps some other players to invest at time two. But as the number of early (and late) investors increase, dynamic increasing differences

[12] In static global games, one typically derives the existence of a unique rationalizable equilibrium. In our dynamic game, we only show the existence of a unique rationalizable outcome as one cannot iteratively delete strategies that prescribe different behavior following out-of-equilibrium information sets.

implies that waiting becomes less desirable. Hence, we can determine a new cutoff vector $\bar{s}^2 = (\bar{s}_1^2, \bar{s}_2^2)$ where both $\bar{s}_t^2$'s are computed such that if $s_1^i = s_1^2$ ($s_2^i = s_2^2$), player i is indifferent between investing and waiting (investing and not investing) given that player i anticipates all other players to invest at time one (two) if their first-period signals are higher than s_1^1 (if they did not invest at time one and if their second-period signals are higher than $1 + \Delta$). Repeating this procedure, we get a decreasing sequence of cutoff vectors. This sequence must converge, as investing is dominated for sufficiently low signals. Furthermore, it must converge to a symmetric switching equilibrium. To see this, note that a player with a signal $\bar{s}_1^\infty$ must be indifferent between investing at time one and waiting knowing that the other players invest at time one (two) if their first-period signal lies above $\bar{s}_1^\infty$ (if they did not invest at time one and if their second-period signal lies above $\bar{s}_2^\infty$). The reason is that it must be optimal for players with higher signals to invest and it can not be strictly optimal for players with a slightly lower first-period signal to invest, by the construction of the sequence. Players with even lower first period signals prefer not to invest if their rivals play according to $(\bar{s}_1^\infty, \bar{s}_2^\infty)$ because they have a lower estimate of the fundamental and expect less players to invest, which makes waiting more desirable. A similar argument ensures that a player with a second-period signal $\bar{s}_2^\infty$ is indifferent between investing and not investing.

For players with sufficiently low signals it is a dominant strategy not to invest, even if they expect all other players to invest. Mirroring the above argument, one can construct an increasing sequence of cutoff vectors below which every player refrains from investing. This sequence also converges to a symmetric switching equilibrium. To complete the proof, we suppose that the iterative elimination from above and below converge to different symmetric switching equilibria and show that this leads to a contradiction. To see the logic underlying the contradiction, observe that a player who receives a signal equal to the first- (second-) period cutoff level is indifferent between investing and waiting (and not investing). Thus, a player who receives a cutoff signal must expect less investment activity in the higher equilibrium as she is more optimistic about the fundamental. For a player who receives a first- or second-period cutoff signal, the expectation of the total number of investors is only a function of $k_1 - k_2$ because all realizations of the fundamental are equally likely. In the higher equilibrium, a player with a signal equal to the second-period cutoff level expects lower investment activity only if less players have already invested in the first period, that is if k_1 is higher relative to k_2. This requires that $k_1 - k_2$ must have a higher value in the higher equilibrium.

But a player with a signal equal to the first-period cutoff level k_1 expects a lower level of investment activity only if k_2 is relatively higher, which ensures that she expects less players to invest in the second period. This implies, however, that $k_1 - k_2$ must have a lower value in the higher equilibrium, which establishes the contradiction. Thus, there exists a unique symmetric switching equilibrium and hence a unique rationalizable outcome.

How would the introduction of social learning affect this uniqueness result? Clearly, if players perfectly observe past investment activity, there exist many strategy profiles for which some states of the world would be perfectly inferred. Equilibrium common knowledge of the state of the world at time two, however, is likely to give rise to multiple continuation equilibria. Similarly, if all players observed the same (public) signal of past investment activity, the results by Hellwig (2002) on static global games suggest the possibility of multiple equilibria. Hellwig also shows, however, that equilibrium in a static global game is unique if the public signal is not too precise relative to the private one. Dasgupta (2001) analyzes a dynamic global game in which players observe past economic activity plus some idiosyncratic noise under the assumption that there are no cohort effects. He proves that if the variance of the idiosyncratic noise is large enough relative to the one of the private signal, his waiting game is characterized by a unique equilibrium within the class of symmetric switching equilibria. One reason he focuses on switching equilibria is that rationalizability imposes "too little" restrictions on the inferences players can draw from past investment activity. A full characterization of the set of equilibrium outcomes in a dynamic global game with social learning and irreversible investment is needed to check the robustness of Proposition 2 to the introduction of social learning. This, however, is beyond the scope of the current paper.

4.2 Characterization of Symmetric Switching Equilibria

In this section, we characterize some symmetric switching equilibria. This allows us to derive comparative statics results in the NC and the PI interpretations of our model. Furthermore, we will use the explicit characterization of equilibria to discuss how the absence of dynamic increasing differences can lead to multiple equilibria in the following subsection.

A necessary condition for a strategy profile k^* in which $k_t^* < \infty$ (for $t = 1, 2$) to be an equilibrium (strategy profile) in symmetric switching strategies is that it satisfies the following two equations:

$$g(k_1^*, k^*) = 0, \tag{4}$$

$$h(k_2^*, k^*) = 0. \tag{5}$$

Equation (4), which can be rewritten as

$$k_1^* + E(n \mid k^*, s_i^1 = k_1^*) - 1 = W(k^*, s_i^1 = k_1^*),$$

states that a player possessing a first-period signal $s_1^i = k_1^*$ must be indifferent between investing and waiting. Equation (5) says that a player who receives a second-period signal $s_2^i = k_2^*$ is indifferent between investing and not investing. In case $k_1^* = \infty$, equation (4) must be replaced by the condition $g(s_1^i, k^*) \leq 0$, for all s_1^i. That is, it must be optimal to refrain from investing for all first period signals. Similarly, in case $k_2^* = \infty$, condition (5) must be replaced by the condition $h(k_2^*, k^*) \leq 0$ for all s_2^i.

If $g(s_1^i, k)$ (respectively $h(s_2^i, k)$) are monotonically increasing in s_1^i (respectively s_2^i), then any strategy profile k^* satisfying (4) and (5) is clearly an equilibrium strategy profile. However, note that $h(\cdot)$ is a function of $E(\gamma n_1 + n_2 \mid \cdot)$. In some symmetric switching equilibria, players refrain from investing for sufficiently low signals and all players invest immediately in the first period for sufficiently high signals. For intermediate signals, however, players wait and invest in the second period when receiving good news. If there are late mover cohort effects ($\gamma < 1$) in such a candidate equilibrium, $h(\cdot)$ need not be monotone in s_2^i as $E(n_2 \mid \cdot)$ is not. When characterizing the set of symmetric switching equilibria, we first look for candidate equilibria that solve equations (4) and (5) and then carefully verify that these candidate equilibria are indeed equilibria. To economize on notation, we will from now on denote equilibrium strategy profiles (and candidate equilibria) by k rather than k^*.

We refer to an equilibrium k in which no player invests in the second period as an *immediate investment equilibrium*. Formally, k is an immediate investment equilibrium if and only if $k_2 \geq k_1 + \epsilon$.

PROPOSITION 3 *There exists an immediate investment equilibrium if and only if* $\Delta \geq -\frac{1}{2} + \epsilon + \frac{3}{4}\gamma$. *In an immediate investment equilibrium* $k_1 = \frac{1}{2}$.

The parameter condition under which an immediate investment equilibrium exists is intuitive. As the payoff reduction for late movers Δ increases, players have an incentive to move early and thus an immediate investment equilibrium is more likely to exist. As γ decreases, a player who deviates in order to invest late enjoys a smaller (inter-period) network effect, which makes deviating less attractive. Hence, as γ decreases, an immediate investment equilibrium is more likely to exist. To understand why an increase in ϵ makes it harder to sustain an immediate investment equilibrium, consider a player with a signal $s_1^i = 1/2$. This player is uncertain about whether the fundamental θ is high enough to make her investment profitable. As ϵ increases, more uncertainty about θ is resolved between period one and two, which makes it more desirable to wait in order to receive more information.

To further understand the role of ϵ, it is useful to note that the expected network benefit for a player with a signal $s_1^i = k_1$ equals $1/2$ in an immediate investment equilibrium.[13] Intuitively, player i knows that all players possessing a signal higher (lower) than hers invest (do not invest) at time one. Player i asks herself the question: What is the mass of players who received a first-period signal greater than k_1? Player i knows that θ lies in a 2ϵ neighborhood of s_1^i. If $\theta > s_1^i$, she knows that more than $1/2$ of the population possesses a signal higher than hers. Conversely, if $\theta < s_1^i$, she knows that more than $1/2$ of the population posses a signal strictly lower than hers. Given hat $\theta|s_1^i$ is symmetrically distributed around s_1^i, player i knows that the event $\theta > s_1^i$ is as likely to occur as the event $\theta < s_1^i$. Therefore $E(n_1|s_1^i = k_1, (k_1, \infty)) = 1/2$. Stated differently, player i always believes to lie in the center of the world. She always expects half of the population to possess a signal strictly higher than hers, with the other half possessing a signal strictly lower than hers.

Now, for the sake of argument, suppose there is no inter-period network effect for late movers ($\gamma = 0$) and that $\Delta = 0$. Then, an immediate investment equilibrium does not exist whenever $\epsilon > 1/2$. The intuition for this result goes as follows: In an immediate investment equilibrium a player with a signal $s_1^i = k_1$ is indifferent between investing and not investing, which is the action she will take if she

[13] Formally, using the $n_1(\theta, k)$ function derived in Appendix 2 and the fact that

$$E(n_1 \mid s_1^i = k_1, k) = \frac{1}{4\epsilon} \int_{k_1 - 2\epsilon}^{k_1 + 2\epsilon} n_1(\theta, k)\, d\theta,$$

it is easy to verify that $E(n_1 \mid s_1^i = k_1, k) = \frac{1}{2}$.

decides to wait. So her expected payoff must be zero. Furthermore, as discussed above, she expects half of the population to get a better signal than herself. So her expected gain from the network effect is 1/2. But if $\epsilon > 1/2$, this player could wait, forfeit the expected network effect and only invest if she learns that she was too pessimistic. In this case her expected payoff when getting good news changes by $\epsilon - 1/2$, while her expected payoff when getting bad news remains zero. So if $\epsilon > 1/2$ this is a profitable deviation and an immediate investment equilibrium cannot exist. In general, the more uncertainty about the fundamental is revealed before the second period, the more attractive it becomes to wait, and the less likely it is that an immediate investment equilibrium exists.

We will refer to an equilibrium in which players with high signals invest immediately and players with intermediate signals wait and invest later when receiving good news - but not when receiving bad news - as an *informative waiting equilibrium.* When solving for informative waiting equilibria, it is convenient to slightly relax the definition of symmetric switching equilibria and solve for all symmetric strategy profiles that can be characterized by a vector (k_1, k_2) with the interpretation that (i) $\sigma^i(s_1^i) = 1$ if and only if $s_1^i > k_1$, and (ii) for all $s_2^i < k_1+\epsilon$, $\sigma(s_1^i, s_2^i) = 1$ if and only if $s_2^i > k_2$. We refer to such equilibria as weak symmetric switching equilibria. The difference to our earlier definition is that we only require switching behavior on the equilibrium path. In the out-of-equilibrium event that a player with signal $s_1^i > k_1$ did not invest and gets a signal $s_2^i > k_1 + \epsilon$, we do not solve for this player's optimal behavior explicitly.[14] Formally, an informative waiting equilibrium is a (weak symmetric switching) equilibrium in which $k_1 - \epsilon < k_2 < k_1 + \epsilon$.

For brevity, let $x \equiv 4\epsilon + \gamma$ and let

$$D \equiv -16\Delta + 16\epsilon - 8 + 12\gamma + [(2-\alpha) - (2-\tau)x]^2,$$

$$\Delta^a \equiv \Delta^b + \frac{1}{16}\,[(2-\alpha) - (2-\tau)x]^2,$$

$$\Delta^b \equiv -\frac{1}{2} + \frac{3}{4}\gamma + \epsilon,$$

$$\Delta^c \equiv \frac{\gamma}{4}(1+\tau) - \frac{(1+\alpha) + 4\epsilon(1-\tau)}{4}.$$

[14]We, nevertheless, require player i's strategy to be sequentially rational, i.e. she must invest in the second period if and only if it is profitable for her to do so. This optimal behavior may, however, require a player with signal $s_2^i > k_2$ not to invest following an out-of equilibrium history in which she did not invest when receiving a signal $s_1^i > k_1$.

We are ready to characterize when an informative waiting equilibrium exists.

Proposition 4 *There exists an informative waiting equilibrium* (k_{11}, k_{21}) *if the following three conditions are satisfied: (a)* $\Delta \leq \Delta^a$*, (b) either* $(2-\alpha) > (2-\tau)x$ *or* $\Delta \leq \Delta^b$*, and (c)* $\Delta > \Delta^c$*. In this informative waiting equilibrium*

$$k_{11} = \frac{1}{8}\ \{\tau(\tau-2)x^2 + 2x[1-(1-\alpha)(1-\tau)] + (1-\alpha)^2 + 3 + (x\tau-\alpha)\sqrt{D}\},$$

$$k_{21} = \frac{1}{8}\{-\sqrt{D}(\alpha+8\epsilon-x\tau)+x^2\tau^2+2\tau x(1-\alpha-4\epsilon-x)+(2-\alpha)^2-8(1-\alpha)\epsilon+2\alpha(1+x)+16\epsilon x\}.$$

Furthermore, there exists an informative waiting equilibrium (k_{12}, k_{22}) *if the following three conditions are satisfied: (a)* $\Delta \leq \Delta^a$*, (d)* $(2-\alpha) > (2-\tau)x$*, and (e)* $\Delta > \Delta^b$*. In this informative waiting equilibrium*

$$k_{12} = \frac{1}{8}\ \{\tau(\tau-2)x^2 + 2x[1-(1-\alpha)(1-\tau)] + (1-\alpha)^2 + 3 - (x\tau-\alpha)\sqrt{D}\},$$

$$k_{22} = \frac{1}{8}\{\sqrt{D}(\alpha+8\epsilon-x\tau)+x^2\tau^2+2\tau x(1-\alpha-4\epsilon-x)+(2-\alpha)^2-8(1-\alpha)\epsilon+2\alpha(1+x)+16\epsilon x\}.$$

Conversely, there exists no other informative waiting equilibrium.

To understand under what conditions an informative investment equilibrium exists, suppose first that $(2-\alpha) < (2-\tau)x$, as is the case if the ex post payoff function satisfies dynamic increasing differences. Then, since condition (d) is violated, the (k_{12}, k_{22}) equilibrium does not exist. Next, observe that in this case conditions (a) and (b) are satisfied whenever Δ is too low to sustain an immediate investment equilibrium, i.e. when waiting to act on more information is profitable. The role of condition (c) is to ensure that the relevant decision for a player with signal $s_1^i = k_1$ is whether to wait for good news or whether to invest immediately. If it is violated, the player would prefer to invest in the second period also when getting bad news (which explains why condition (c) gives a lower bound on Δ). Condition (c) is always satisfied when the ex post payoff function exhibits dynamic increasing differences. Whenever the ex post payoff function satisfies dynamic increasing differences, the only reason to wait is to collect information in order to make a better informed decision. So if a player would prefer to invest when getting bad news, she could invest immediately and save the waiting costs. If cohort effects are such that dynamic increasing differences are violated, however, one may want to wait and invest both when getting good news and when getting bad news. In this case a player waits in order to benefit from a higher network effect. Note that this requires γ to be sufficiently greater than α; i.e. late movers must enjoy a higher inter-period network effect than the early movers. The intuition for this is that a player with signal $s_1^i = k_1$ expects half of the population to invest in the first period. So she can expect at most half of the population to invest late. Therefore, she can only expect to gain a larger network effect by moving late if the inter-period network effect for late movers γ is greater than its first period counterpart α.

We are left to consider the case in which $(2-\alpha) > (2-\tau)x$. Trivially, this implies that conditions (b) and (d) are satisfied. Clearly, then the equilibrium (k_{11}, k_{21}) exists for all $\Delta \in (\Delta^c, \Delta^a]$ and the equilibrium (k_{21}, k_{22}) exists for all Δ in the nonempty interval $(\Delta^b, \Delta^a]$. Since an immediate investment equilibrium exists for all $\Delta > \Delta^b$, this implies that *if* $(2-\alpha) > (2-\tau)x$, *there exist values of* Δ *for which our model has multiple equilibria* as long as $\Delta^a \geq 0$. Observe that a necessary - though not sufficient - condition for $(2-\alpha) > (2-\tau)x$ is that the ex post payoff function violates our condition of dynamic increasing differences. As either cohort effects increase (i.e. α or γ decrease), the condition is more likely to be satisfied. One interpretation of this fact is that as cohort effects become more important, dynamic coordination becomes more important. A player then only wants to invest if she believes that the other players invest at the same point in time. Second, if $\tau < 1$, then decreasing γ is more likely to make this condition hold than decreasing α by the same amount, reflecting the fact that a first-period decision maker discounts the second-period investment payoffs. Third, as the uncertainty ϵ increases, the condition is less likely to hold. Intuitively, as ϵ increases more uncertainty about the fundamental is revealed before the second period. For a player who is unsure about whether she should invest in the first period, it is thus more profitable to wait for additional news and relatively less important to invest when the other players invest. As the coordination aspect becomes relatively less important, multiple equilibria are less likely to exist.

Note that our game is characterized by multiple equilibria *despite* the fact that (i) we only focus on symmetric switching equilibria, (ii) we work with a uniform prior along the entire real line, and (iii) there is no social learning in our model. Hence, the absence of social learning allows us to identify cohort effects as the driving force behind our multiplicity result.

The following Lemma rules out the existence of other (weak) symmetric switching equilibria in all of the economic environments discussed in Section 3.

LEMMA 1 *If* $\tau < 1$ *or if* $\alpha = \gamma = \tau = 1$ *and* $\Delta > 0$, *then there exists no (weak) symmetric switching equilibrium in which* $k_1 = \infty$. *Furthermore, if* $\frac{1}{2}(\tau\gamma - \alpha) - \epsilon(1 - \tau) < \Delta$ *then there exists no equilibrium in the class of (weak) symmetric switching equilibria in which* $k_2 < k_1 - \epsilon < \infty$.

The first set of conditions rule out an equilibrium in which players only invest in the second period. If $\tau < 1$ players discount the payoffs of investing in the second period as their benefits from investing are delayed. In this case, as the fundamental θ increases without bound, the foregone first-period benefit grows without bound. In other words, if $\tau < 1$, then it is a dominant strategy to invest in the first period for sufficiently high signals. Thus, there cannot exist an equilibrium in which players only invest in the second period. In the PI case considered, a player with a very high second period signal has a strict incentive to invest. Hence, a player with a very high first period signal realizes that she will want to invest in the second-period independent of whether she gets good or bad news. But then she is better off investing at time one and saving the waiting cost Δ.

The second condition ensures that a symmetric switching equilibrium in which players wait in order to invest in the second period with certainty fails to exist. This type of behavior rules out any informational reason for waiting. Rather waiting must be driven by the desire to coordinate the timing of the investment. This can only be profitable if there are cohort effects and if, as discussed above, the second-period cohort effect is sufficiently less than the first-period cohort effect (i.e. $\gamma > \alpha$).

4.3 Cohort Effects and Multiplicity

In this subsection, we provide illustrative examples to discuss why the lack of dynamic increasing differences can lead to multiple equilibria. These examples help in distinguishing the underlying force behind our multiple equilibrium phenomena from that found in other dynamic global game models. We then observe, however, that even though the FL interpretation of our model always violates dynamic increasing differences, it has a unique equilibrium within the class of symmetric switching strategies for a wide range of parameter values.

Example 1. Consider the FL game with a discount factor of $\delta = 3/5$ and let $\epsilon = 1/64$. Using the normalization introduced in Section 3, one has $\tau = 3/5$, $\alpha = 3/8$, $\gamma = 5/8$ and $\Delta = 0$. Since $\Delta > \Delta^b = -1/64$ in this case, Proposition 3 implies that there exists an immediate investment equilibrium. Furthermore, as in addition $\Delta < \Delta^a \approx 0.046$ and $(2-\alpha) > (2-\tau)x$, Proposition 4 implies that there exists two informative waiting equilibria in this example.

Indeed, in the above example there exist multiple continuation equilibria for some first-period cutoff levels. To see this, consider the second-period continuation game of the immediate investment equilibrium; i.e. suppose it is common knowledge that all players invested in the first-period if and only if they received a signal $s_1^i > 1/2$. Obviously, there exists a continuation equilibrium in which no player invests as otherwise the immediate investment equilibrium would not exist. We now show that there exists another continuation equilibrium in which an active player invests if and only if she receives a signal $s_2^i > 1/2$. Consider first a player whose $s_2^i = \frac{1}{2}$. Using Lemma 3 (which can be found in the Appendix) it is easy to verify that $E_2(n_1 \mid 1/2, (1/2, 1/2)) = 1/2$ and that $E_2(n_2 \mid 1/2, (1/2, 1/2)) = 3/16$. Player i's gain of investing, given her anticipation that all active players with a $s_2^i \in [\frac{1}{2}, \frac{1}{2}+\epsilon]$ invest at time two, equals

$$h(\frac{1}{2}, (\frac{1}{2}, \frac{1}{2})) = \frac{1}{2} + \frac{5}{8}\frac{1}{2} + \frac{3}{16} - 1 = 0.$$

We next show that $\frac{\partial h(s_2^i, (\frac{1}{2}, \frac{1}{2}))}{\partial s_2^i} > 0$ for all $s_2^i \in [1/2, 1/2+\epsilon]$, thereby proving that it is optimal to invest for all signals above 1/2 in the continuation game if all other players follow the same cutoff strategy. One has

$$\frac{\partial h(s_2^i, (\frac{1}{2}, \frac{1}{2}))}{\partial s_2^i} = 1 + \frac{\gamma}{2\epsilon}[n_1(s_2^i + \epsilon, (\frac{1}{2}, \frac{1}{2})) - n_1(s_2^i - \epsilon, (\frac{1}{2}, \frac{1}{2}))]$$
$$+ \frac{1}{2\epsilon}[n_2(s_2^i + \epsilon, (\frac{1}{2}, \frac{1}{2})) - n_2(s_2^i - \epsilon, (\frac{1}{2}, \frac{1}{2}))]$$

Using Lemma 3 to substitute all relevant $n_1(\cdot)$'s and $n_2(\cdot)$'s in the above equation, the reader can check that

$$\frac{\partial h(s_2^i, (\frac{1}{2}, \frac{1}{2}))}{\partial s_2^i} \geq 1 + \frac{2\gamma - 1}{8\epsilon} > 0 \qquad \forall s_2^i \in [1/2, 1/2+\epsilon].$$

Hence, there also exists a continuation equilibrium in which all active players with a $s_2^i \in [1/2, 1/2+\epsilon]$ invest at time two.

Thus, in Example 1 the continuation game is not a global game. This bears some resemblance with the models of Angeletos, Hellwig and Pavan (2003) and Chamley (1999). Nevertheless, the multiplicity is driven by another force. In the papers mentioned above, all players observe a public signal, which informs them about the state of the world. For example in Angeletos, Hellwig and Pavan players observe that a devaluation has not occurred. From this, they can deduce in equilibrium that devaluing is never a dominant strategy for the central bank. Hence, attacking is never a dominant strategy in the continuation game. In Chamley, players draw inferences about the state of the world based on the outcome induced by past aggregate behavior and their private signals. As the observation of past behavior is shared by all agents, it also acts as a public signal. In particular, if the observation becomes too informative about the state of the world, then Chamley's model has multiple equilibria.[15] Because aggregate past behavior is not observed in our model, the above does not drive the multiplicity result in our model. Indeed, if investments were perfectly reversible as in Angeletos, Hellwig and Pavan, our model would predict a unique rationalizable outcome.[16] Similarly, if there were no endogenous timing decision, as in Chamley's model, we would obtain a unique rationalizable outcome.

The multiplicity in our model comes from the fact that with irreversible investment and strong cohort effects, players have an incentive to invest when other players invest. This gives rise to a coordination problem regarding the timing of investment. Example 2 further illustrates this timing problem.

Example 2. $\gamma = 1$; $\alpha = \frac{1}{2}$; $\epsilon = \frac{1}{16}$; $\tau = 0.99$ and $\Delta = 0.315$.

In this example

$$\Delta^b = \frac{1}{4} + \frac{1}{16} = 0.3125 < \Delta = 0.315 < \Delta^a = \Delta^b + \frac{1}{16}((2-\alpha) - (2-\tau)x)^2 = 0.316,$$

and

$$(2-\alpha) - (2-\tau)x \simeq 0.24 > 0.$$

[15]See the discussion in the Introduction.

[16]To continue to abstract from social learning, one would need to assume that the first-period investment payoffs are only observed after the second-period investment decision.

Hence, from Propositions 3 and 4 we know that our example is characterized by multiple equilibria.

As $\gamma = 1$, if players follow a cutoff strategy in the first period, then in Example 2 the continuation game is always a well defined global game. More precisely, fix any k_1 and construct an induced game in which nature invests on behalf of a player in the first period if and only if the player receives a signal $s_1^i > k_1$ and which is otherwise identical to the above example. We now argue that this induced game is always solvable through iterative elimination of dominated strategy. Clearly it is not optimal to invest for very low signals. Iteratively eliminating strategies from below, there will be a unique level $\underline{s}_2^\infty$ below which it is optimal to refrain from investing. In case $\underline{s}_2^\infty \geq k_1 + \epsilon$, iterative elimination of dominated strategies implies that no player wants to invest in the second period. Hence, consider the case in which $\underline{s}_2^\infty < k_1 + \epsilon$. In this case, a player who receives a signal $s_2^i = \underline{s}_2^\infty$ is indifferent between investing and not investing if she anticipates all other active players with a higher second-period signal to invest. As $\gamma = 1$, the profitability of investing late only depends on the state of the world and on the total number of investors in either period. Now consider a player with a higher second-period signal who anticipates all other active players with a higher signal than hers to invest. This player is thus more optimistic about the state of the world and must (weakly) expect more players to invest in either period. Hence, she strictly prefers to invest. In particular, for a player with signal $s_2^i = k_1 + \epsilon$, the condition that all active players with a higher signal than hers invest, is vacuous and therefore she strictly prefers to invest. Iterative eliminating strategies from above, hence, implies that all players with a signal above $s_2^i = \underline{s}_2^\infty$ have a strict incentive to invest.

Why then does Example 2 have multiple equilibria? The reason is that as $\alpha < \tau$ dynamic increasing differences are violated and a player who anticipates that more players wait and invest at time two, has a higher incentive to do the same. On the other hand, if she thinks that the other investors are more likely to invest at time one, waiting becomes less attractive. Hence, the lack of dynamic increasing differences gives rise to a coordination problem regarding the timing of investment.

Observe that in both examples, for sufficiently high signals, all undominated strategies must prescribe player i to invest at time one. Similarly, for sufficiently low signals, all undominated strategies must prescribe player i to never invest. There are, however, no signals for which a player - independently of her rivals' strategies

- prefers to wait and to invest at time two. This raises the following question: "Is our multiplicity result not fundamentally driven by an insufficiently rich signal structure?" The reasoning behind the question is as follows: In the presence of cohort effects, players face a two-dimensional coordination problem; first, they must decide whether to invest or not, and, in case they decide to invest, they must choose their investment period. Signals, however, are only one-dimensional and one cannot apply a process of iterative deletion of dominated strategies to the second dimension of our coordination problem as there is no player for whom it is dominant to invest at time 2. This reasoning, however, is incomplete.[17] The global game literature derives its results through iterative elimination of dominated strategies — exploiting the fact that players have increasing best replies.[18] Admittedly, increasing best replies are not a necessary condition for the introduction of noise to reduce the set of equilibria (see, for example, Guesnerie (1992) and Mason and Valentinyi (2003)), but without them there exists no general results and there can be no presumption that equilibrium would be unique if one added another signal as to when to invest. To fully address this question, however, one would need to solve a dynamic global game with multiple signals. To the best of our knowledge, there exists no paper that investigates the impact of multiple signals in any global game setting, and such an extension is beyond the scope of the paper.

Finally, observe that the lack of dynamic increasing differences does not imply that the coordination problem regarding the timing of investment is severe enough to give rise to multiple equilibria. For example, if Δ is very high then investing at time one always dominates investing at time two and our game has a unique equilibrium. More interestingly, the following proposition gives necessary and sufficient conditions for the existence of a unique symmetric switching strategy equilibrium in the FL interpretation of our model. The proof follows immediately from the results of the previous subsection and is thus omitted.

PROPOSITION 5 *Consider the adopting a technology with a fixed lifespan (FL) interpretation of our model. In this interpretation, there exists multiple equilibria if and only if the following holds:*

$$-\frac{1}{2}+\epsilon+\frac{3}{4}\gamma<0<-\frac{1}{2}+\epsilon+\frac{3}{4}\gamma+\frac{1}{16}[(2-\delta\gamma)-(2-\delta)(\gamma+4\epsilon)]^2,$$

[17] Recall also that if the game satisfies dynamic increasing differences, then there exists a unique equilibrium even though there exists no signal for which it is dominant to invest at time 2.

[18] See for example Frankel, Morris and Pauzner (2003).

$$(2-\delta\gamma) > (2-\delta)(\gamma+4\epsilon).$$

One implication of the above proposition is that for any given level of setup costs s and any given discount factor δ, if the technology lasts a sufficiently large number of periods T, then the FL model has a unique symmetric switching equilibrium. The technical reason is that as T increases, γ approaches 1 and both $0 < -\frac{1}{2}+\epsilon+\frac{3}{4}\gamma$ and $(2-\delta\gamma) < (2-\delta)(\gamma+4\epsilon)$. Economically, the late mover cohort effect becomes less and less important and thus the dynamic coordination issue becomes less relevant. Hence, even though dynamic increasing differences are only satisfied in the limit as $T \to \infty$, there exists a unique equilibrium within the class of symmetric switching strategies for large enough T.

Similarly, if players discount the future heavily, i.e. if δ is very low, γ becomes large and there exists a unique symmetric switching equilibrium. Intuitively, as δ goes to zero, players strongly discount the future and hence the belief that many investors move late does not suffice to encourage a player to postpone her investment decision. Again the dynamic coordination aspect becomes too small to sustain multiple equilibria. Furthermore, as ϵ, which measures the uncertainty about the fundamental that is resolved between the first and the second period, becomes sufficiently large, there exists a unique equilibrium. Intuitively, the belief that many investors invest early will have less of an impact because it is more important to wait and gather information about the true fundamental. In this sense, the dynamic coordination aspect becomes relatively less important and a unique symmetric switching equilibrium is more likely to exist.

5 Comparative Statics

In this section we restrict ourselves to the case in which the ex post payoff function satisfies dynamic increasing differences. We use the closed form solutions, which we computed in subsection 4.2, of our essentially unique equilibrium to establish some comparative static results.

We start with the following technical clarification. If one takes the improper prior literally, then the probability that θ lies in any finite interval is zero and hence with probability one players have a dominant strategy. Nothing in our analysis hinges on the fact that the prior is distributed over the entire real line and one could assume instead that it is distributed $U[-\overline{\theta},\overline{\theta}]$ for sufficiently large $\overline{\theta}$, which we will do implicitly when deriving our comparative static results. More seriously, we will derive comparative static results on parameters of the underlying economic interpretations given in Section 3. When doing these comparative statics, we fix the distribution of $\tilde{\theta}$ and ask how investment activity is affected by changes in the underlying parameters. It is thus important to realize that a change in one of those parameters does not only affect the equilibrium cutoff levels but it also affects the value of θ, the normalized realization of the fundamental.

We first tackle our PI interpretation. Recall that Δ represents effective discount offered to time-one investors relative to time-two investors. Consider a government that taxes the investment from late adopters only. The following proposition implies that a higher tax will increase the speed with which players commit but will not affect the expected total investment activity.

PROPOSITION 6 *Consider the PI interpretation. Expected first-period investment activity strictly increases in* Δ *over* $[0, \Delta^b]$ *and remains constant thereafter. Expected second period investment activity strictly falls in* Δ *over* $[0, \Delta^b]$ *and remains zero thereafter. Total expected investment activity, however, is independent of* Δ.

Next, we focus on the NC interpretation of our model.

PROPOSITION 7 *Consider the NC interpretation.*
1) If the upfront cost of becoming a member (F) *increases, expected first-period membership decreases, while the expected mass of new members that join in period two remains constant. Thus, the overall membership decreases in* F.
2) The expected mass of new members that join in period two is increasing in the discount factor δ. *The expected mass of players that join in period 1 can increase or decrease in* δ.
3) If $F > 0$, *expected first-period membership is increasing in* δ *for sufficiently small* ϵ. *Furthermore, the overall expected membership* $E(n_1 + n_2|k)$ *is increasing in* δ *for sufficiently small* ϵ. *The positive effect of increasing* δ *on the expected mass of players that join in period two vanishes as* ϵ *approaches zero.*

Both from a first and a second period perspective, an increase of the up-front payment F reduces the attractiveness of the club. A player is thus only willing to join in the first (or second) period if she expects a better state of the world. Consider, however, a player who is indifferent between investing in the first period and waiting. If she waits and gets good news, her assessment about the quality of the club increases by ϵ in either scenario. Similarly, in either scenario, her beliefs about the expected network benefit from first period adopters increase by the same amount. In this sense, the intertemporal tradeoff and thus the decision whether or not to wait is not affected by F. F simply shifts the critical point at which players are willing to invest in either period by the same amount.

An increase in δ has various effects. First, in the presence of an up-front payment it becomes more attractive to join the club because future rewards are discounted less. It also, however, affects the intertemporal trade-off since the gain of waiting and taking a better informed decision is discounted less while the cost - foregone first period profits - is not affected. Both effects tend to increase the expected mass of players that join in the second period. From a first period perspective, however, these effects work in opposite directions. In addition, if more player join in period two, the network effect implies that joining early becomes more profitable. Indeed, depending on the exact parameter values, expected first period investment activity can either fall or increase. Second period investment activity, however, always increases.

As ϵ goes to zero, the information a player gets about the fundamental when waiting becomes negligible. Nevertheless, in equilibrium, there is considerable informative value from waiting because a player can better predict her rival's investment behavior. If, however, the only informative value concerns the equilibrium behavior of rivals, the above proposition shows that a higher discount factor raises overall investment activity. Surprisingly, the increase in the expected mass of members that join in the second period becomes negligible as ϵ goes to zero, while first period investment activity increases.

6 Conclusion

We investigated irreversible investment decisions with positive network effects using a dynamic global game approach. In contrast to most papers on global games, we did not focus on determining conditions on the prior distribution and signaling distribution that give rise to a unique equilibrium. Instead, we used a Laplacian prior, a simple signaling technology, and abstracted from social learning. This allowed us to focus on the interaction between positive network effects and irreversible investments. We showed that with irreversible investments positive contemporaneous network effects do not necessarily imply dynamic increasing differences.

Using this fact, we illustrated that in a dynamic setting the global game approach may not give rise to a unique prediction. If dynamic increasing differences are violated a dynamic coordination aspect arises: Players have an incentive to invest at the same time others do. If this dynamic coordination aspect is strong enough, our global game has multiple equilibria. Nevertheless, we found the global game approach to be powerful. Even in the fixed lifespan technology adoption problem (FL) that always failed dynamic increasing differences, we could provide weaker conditions for the existence of a unique equilibrium within the class of symmetric switching strategies. Moreover, we could show that there are at most three such equilibria in the FL interpretation.

Our results highlight that the standard assumption that network benefits only depend on the total number of investors and not on when others invest can have strong consequences if investments are irreversible. Even in the financial sector, reversing ones investment decision is typically costly. We believe an interesting question for future research is how big the impact of such transaction costs are in dynamic models of speculative attacks or other macroeconomic coordination failures. In our model cohort effects rest a.o. on technological factors. We believe that cohort effects may also arise in other contexts due to different reasons: For instance a successful speculative attack is more likely to occur when all speculators attack the currency at the same time than if they were to attack the peg at different moments in time. Future research may also shed some light on the nature and causes of cohort effects in different economic environments.

Appendix

We start by establishing a few Lemmas and by introducing some notation.

LEMMA 2 $E(n_j|s_1^i,k) = \frac{1}{2}E(n_j|s_2^i = s_1^i + \epsilon, k) + \frac{1}{2}E(n_j|s_2^i = s_1^i - \epsilon, k)\ \forall j = 1, 2.$

Proof: Trivially, one has

$$E(n_j|s_1^i,k) = \frac{1}{4\epsilon}\int_{s_1^i-2\epsilon}^{s_1^i+2\epsilon} n_j(\theta,k)\ d\theta$$

$$= \frac{1}{2}\{\frac{1}{2\epsilon}\int_{s_1^i-2\epsilon}^{s_1^i} n_j(\theta,k)\ d\theta + \frac{1}{2\epsilon}\int_{s_1^i}^{s_1^i+2\epsilon} n_j(\theta,k)\ d\theta\}$$

.

$$= \frac{1}{2}E(n_j|s_2^i = s_1^i + \epsilon, k) + \frac{1}{2}E(n_j|s_2^i = s_1^i - \epsilon, k).$$

Q.E.D.

LEMMA 3 *One has*

$$n_1(\theta,k) = \begin{cases} 0 & \textit{if } \theta < k_1 - 2\epsilon, \\ \frac{2\epsilon+\theta-k_1}{4\epsilon} & \textit{if } k_1 - 2\epsilon \leq \theta < k_1 + 2\epsilon, \\ 1 & \textit{if } k_1 + 2\epsilon \leq \theta, \end{cases}$$

and $\forall k_2 \in (k_1 - \epsilon, k_1 + \epsilon)$, *one has*

$$n_2(\theta,k) = \begin{cases} 0 & \textit{if } \theta < k_2 - \epsilon, \\ \frac{\epsilon-k_2+\theta}{4\epsilon} & \textit{if } k_2 - \epsilon \leq \theta < k_1, \\ \frac{k_1+\epsilon-k_2}{4\epsilon} & \textit{if } k_1 \leq \theta < k_2 + \epsilon, \\ \frac{k_1+2\epsilon-\theta}{4\epsilon} & \textit{if } k_2 + \epsilon \leq \theta < k_1 + 2\epsilon, \\ 0 & \textit{if } k_1 + 2\epsilon \leq \theta. \end{cases}$$

Proof: To compute $n_1(\cdot)$ and $n_2(\cdot)$ we will work with the following figure and variants thereof:

[Insert Figure 2 here]

In the above figure,the two thick black lines represent all the possible realizations that $(\epsilon_2^i, \epsilon_1^i)$ can take. For example, player a in the graph received a $\epsilon_1^i = \epsilon_2^i = -\epsilon$, while player d received $\epsilon_1^i = \epsilon_2^i = \epsilon$. All the players who received a $\epsilon_1^i = -\epsilon$ are situated on the lower thick line. Those players will receive good news at time two. Similarly, all players situated on the upper thick line will receive bad news at time two. Actually, one can best think of the graph above as possessing a third dimension representing $f((s_1^i, s_2^i)|\theta)$. From above, we know that $\Pr(\epsilon_1^i = \epsilon) = \frac{1}{2}$, that ϵ_2^i is independent of ϵ_1^i and that ϵ_2^i is drawn from a uniform distribution. Hence, we know that half of our population receive an $\epsilon_1^i = -\epsilon$ and lie, uniformly distributed, on the lower thick black line, while the other half lie, uniformly distributed, on the upper thick black line. Therefore, this third dimension is "trivial" and is not shown in the figure.

The diagonal "$k_1 - \theta$" represents the combination of all $(\epsilon_2^i, \epsilon_1^i)$ such that $\epsilon_1^i + \epsilon_2^i = k_1 - \theta$. All players who lie to its right and above possess a first-period signal $s_1^i > k_1$, since $s_1^i = \theta + \epsilon_1^i + \epsilon_2^i \geq k_1$ if and only if $\epsilon_1^i + \epsilon_2^i \geq k_1 - \theta$. Hence, the diagonal $k_1 - \theta$ permits us to compute the mass of period one investors.

e denotes the point in which the diagonal $k_1 - \theta$ cuts the upper thick black line. What are the coordinates of point e? We know that all points on the diagonal satisfy the restriction that their x and y coordinates sum up to $k_1 - \theta$. We also know that in point e the y coordinate equals $+\epsilon$. Therefore the coordinates of point e are $(k_1 - \theta - \epsilon, \epsilon)$. If $k_1 = \theta$, then the diagonal goes through the points b and c (in both points $k_1 - \theta = \theta - \theta = 0 = \epsilon - \epsilon$). If $k_1 = \theta - 2\epsilon$, then the diagonal goes through the point a. This is because in the point a, $k_1 - \theta = -2\epsilon$. Similarly, if $k_1 = \theta + 2\epsilon$, then the diagonal goes through the point d. By continuity, if $\theta - 2\epsilon < k_1 < \theta$, the diagonal $k_1 - \theta$ cuts the thick line situated on the X-axis. Similarly, if $\theta < k_1 < \theta + 2\epsilon$, the diagonal cuts the upper thick line.

The vertical "$k_2 - \theta$" permits us to compute the mass of players who invest at time two. For example, in Figure 2 all players situated on the X-axis and to the right of $k_2 - \theta$ invest at time two. The reason is that an active player having received an $\epsilon_1^i = -\epsilon$ invests at time two if and only if $s_2^i = \theta + \epsilon_2^i > k_2$ or if and only if $\epsilon_2^i > k_2 - \theta$. f denotes the point in which the vertical $k_2 - \theta$ cuts the X-axis.

When doesn't the vertical $k_2 - \theta$ cross the lower thick line? $k_2 - \theta > \epsilon$ if and only if $k_2 - \epsilon > \theta$. This is intuitive: If θ is "low", then no player who received good news will invest at time two. In that case point f lies to the right of point b. Similarly, $k_2 - \theta < -\epsilon$ if and only if $k_2 + \epsilon < \theta$. In hat case point f lies to the left of point a.

When computing $n_2(\cdot)$, we focus on an equilibrium in which $k_1 - \epsilon < k_2 < k_1 + \epsilon$. This implies that

$$k_1 - 2\epsilon < k_2 - \epsilon < k_1 < k_2 + \epsilon < k_1 + 2\epsilon.$$

Therefore we must consider the following six cases: (i) $\theta < k_1 - 2\epsilon$, (ii) $k_1 - 2\epsilon < \theta < k_2 - \epsilon$, (iii) $k_2 - \epsilon < \theta < k_1$, (iv) $k_1 < \theta < k_2 + \epsilon$, (v) $k_2 + \epsilon < \theta < k_1 + 2\epsilon$, and (vi) $k_1 + 2\epsilon < \theta$.

In case (i) we know that $\theta < k_1 - 2\epsilon < k_2 - \epsilon$. From above, we know that this implies that points e and f lie to the right of (respectively) d and b. Hence, $n_1(\theta < k_1 - 2\epsilon, k) = n_2(\theta < k_1 - 2\epsilon, k) = 0$.

In case (ii) we know that point f lies to the right of point b, implying that - due to a low θ - $n_2(k_1 2\epsilon < \theta < k_2 \epsilon, k) = 0$. Moreover we also know that in this case $\theta < k_1$ which implies that the diagonal $k_1 - \theta$ cuts the upper thick line. This case is represented in Figure 3.

[Insert Figure 3 here]

In this case all players situated between points e and d invest at time one. Hence, it is straightforward to compute that $n_1(k_1 - 2\epsilon < \theta < k_2 - \epsilon, k) = \frac{2\epsilon + \theta - k_1}{4\epsilon}$.

In case (iii) θ is still strictly lower than k_1 but the vertical $k_2 - \theta$ crosses the two thick black lines. This case is represented in Figure 2. The coordinates of e are $(k_1 - \theta - \epsilon, \epsilon)$ and the ones of point f' are $(k_2 - \theta, \epsilon)$. We are focusing on an equilibrium in which $k_2 > k_1 - \epsilon$. This last inequality can be rewritten as $k_2 - \theta > k_1 - \theta - \epsilon$ which amounts to stating that point f' always lies to the right of point e. From above we thus know that $n_1(k_2 - \epsilon < \theta < k_1, k) = \frac{2\epsilon + \theta - k_1}{4\epsilon}$. All players lying between $[f, b]$ invest at time two. Hence, $n_2(k_2 - \epsilon < \theta < k_1, k) = \frac{\epsilon - k_2 + \theta}{4\epsilon}$.

In case (iv), θ is higher than k_1. This implies that the diagonal $k_1 - \theta$ cuts the lower thick line. Therefore all players who received an $\epsilon_1^i = \epsilon$ (and who are thus situated on the upper thick line) invest at time one. This case is represented in Figure 4.

[Insert Figure 4 here]

From above we know that the coordinates of point e are $(k_1 - \theta + \epsilon, -\epsilon)$. The coordinates of point f are $(k_2 - \theta, -\epsilon)$. Note that $k_2 - \theta < k_1 - \theta + \epsilon$ if and only if $k_2 < k_1 + \epsilon$. As we work here under the assumption that $k_2 < k_1 + \epsilon$, it follows that point f lies to the left of point e. From the figure it should be clear that $n_1(k_1 < \theta < k_2 + \epsilon, k) = \frac{1}{2} + \frac{\epsilon - k_1 + \theta - \epsilon}{4\epsilon} = \frac{2\epsilon + \theta - k_1}{4\epsilon}$ and that $n_2(k_1 < \theta < k_2 + \epsilon, k)$ $= \frac{k_1 - \theta + \epsilon}{4\epsilon} - \frac{k_2 - \theta}{4\epsilon} = \frac{k_1 + \epsilon - k_2}{4\epsilon}$.

In case (v) point f lies to the left of point a. From above it should be clear that $n_1(k_2 + \epsilon < \theta < k_1 + 2\epsilon, k) = \frac{2\epsilon + \theta - k_1}{4\epsilon}$, and that $n_2(k_2 + \epsilon < \theta < k_1 + 2\epsilon, k) = \frac{k_1 + 2\epsilon - \theta}{4\epsilon}$.

In case (vi) point e (see Figure 3) lies to the left of point a. Therefore $n_1(k_1 + 2\epsilon < \theta, k) = 1$ and $n_2(k_1 + 2\epsilon < \theta, k) = 0$. Q.E.D.

LEMMA 4 *For any k that solves equations (4) and (5) and for which $k_2 \in (k_1 - \epsilon, k_1 + \epsilon)$, one has $h(s_2^i, k) < 0$ if $s_2^i < k_2$ and $h(s_2^i, k) > 0$ if $s_2^i \in (k_2, k_1 + \epsilon)$.*

Proof: Since

$$h(s_2^i, k) = s_2^i + \int_{s_2^i - \epsilon}^{s_2^i + \epsilon} [\gamma n_1(\theta, k) + n_2(\theta, k)] d\theta - 1 - \Delta,$$

Leibnitz's rule implies that

$$\frac{\partial h(s_2^i, k)}{\partial s_2^i} = 1 + [\gamma n_1(s_2^i + \epsilon, k) + n_2(s_2^i + \epsilon, k) - (\gamma n_1(s_2^i - \epsilon, k) + n_2(s_2^i - \epsilon, k))].$$

We have shown in Lemma 3 that $n_1(\cdot)$ is weakly increasing in θ and therefore a sufficient condition for $h(\cdot)$ to be strictly increasing is that

$$n_2(s_2^i + \epsilon, k) \geq n_2(s_2^i - \epsilon, k).$$

By Lemma 3, $n_2(\cdot)$ is weakly increasing in θ for all $\theta \leq k_2 + \epsilon$ and hence $h(s_2^i, k)$ is a strictly increasing function in s_2^i for all $s_2^i \leq k_2$. Since k solves the equations

(4) and (5), $h(k_2, k) = 0$ and we conclude that $h(s_2^i, k) < 0$ if $s_2^i < k_2$.

Next, consider $s_2^i \in (k_2, k_1 + \epsilon)$. Since $h(k_2, k) = 0$, one can rewrite $h(s_2^i, k)$ as

$$h(s_2^i, k) = (s_2^i - k_2) + \gamma[E(n_1 \mid s_2^i, k) - E(n_1 \mid s_2^i = k_2, k)] + [E(n_2 \mid s_2^i, k) - E(n_2 \mid s_2^i = k_2, k)].$$

As $s_2^i > k_2$, the first term is positive. Since, by Lemma 3, $n_1(\theta, k)$ is weakly increasing in θ, Leibnitz's rule implies that $E(n_1 \mid s_2^i, k)$ is weakly increasing in s_2^i. Hence $[E(n_1 \mid s_2^i, k) - E(n_1 \mid s_2^i = k_2, k)] \geq 0$. Thus a sufficient condition for $h(s_2^i, k) > 0$ is that

$$[E(n_2 \mid s_2^i, k) - E(n_2 \mid s_2^i = k_2, k)] \geq 0. \tag{6}$$

To prove that condition (6) is satisfied, we establish below that (i) $E(n_2 \mid s_2^i, k)$ is a concave function in s_2^i for all $s_2^i \in (k_2, k_1 + \epsilon)$, and that (ii) $E(n_2 \mid s_2^i = k_1 + \epsilon, k) = E(n_2 \mid s_2^i = k_2, k)$. By Leibnitz's rule,

$$\frac{\partial E(n_2 \mid s_2^i, k)}{\partial s_2^i} = \frac{1}{2\epsilon}[n_2(s_2^i + \epsilon, k) - n_2(s_2^i - \epsilon, k)],$$

and thus

$$\frac{\partial^2 E(n_2 \mid s_2^i, k)}{\partial (s_2^i)^2} = \frac{1}{2\epsilon}[\frac{\partial n_2(s_2^i + \epsilon, k)}{\partial s_2^i} - \frac{\partial n_2(s_2^i - \epsilon, k)}{\partial s_2^i}].$$

Using the facts that $k_2 + \epsilon < s_2^i + \epsilon < k_1 + 2\epsilon$, $k_2 - \epsilon < s_2^i - \epsilon < k_1$, and Lemma 3, it is easy to check that $\frac{\partial^2 E(n_1 \mid s_2^i, k)}{\partial (s_2^i)^2} = -\frac{1}{4\epsilon^2}$.

We are left to show that $E(n_2 \mid s_2^i = k_1 + \epsilon, k) = E(n_2 \mid s_2^i = k_2, k)$. Using Lemma 3, one has

$$E(n_2 \mid s_2^i = k_1 + \epsilon, k) = \frac{1}{2\epsilon}\int_{k_1}^{k_2+\epsilon} \frac{k_1 + \epsilon - k_2}{4\epsilon} d\theta + \frac{1}{2\epsilon}\int_{k_2+\epsilon}^{k_1+2\epsilon} \frac{2\epsilon + k_1 - \theta}{4\epsilon} d\theta,$$

and

$$E(n_2 \mid s_2^i = k_2, k) = \frac{1}{2\epsilon}\int_{k_2-\epsilon}^{k_1} \frac{\epsilon - k_2 + \theta}{4\epsilon} d\theta + \frac{1}{2\epsilon}\int_{k_1}^{k_2+\epsilon} \frac{k_1 + \epsilon - k_2}{4\epsilon} d\theta.$$

Thus

$$E(n_2 \mid s_2^i = k_1 + \epsilon, k) - E(n_2 \mid s_2^i = k_2, k) = \frac{1}{8\epsilon^2}[\int_{k_2+\epsilon}^{k_1+2\epsilon} (2\epsilon + k_1 - \theta) d\theta - \int_{k_2-\epsilon}^{k_1} (\epsilon - k_2 + \theta) d\theta].$$

Integrating this last expression shows that $E(n_2 \mid s_2^i = k_1 + \epsilon, k) - E(n_2 \mid s_2^i = k_2, k) = 0$. Q.E.D.

Let Σ^0 be the set of all strategies. Let Σ^n be the set of all strategies that are undominated after n rounds of iterative elimination of dominated strategies. Let $\sigma^n \in \Sigma^n$. Let $\underline{s}_t^n(\sigma^n)$ be the supremum below which σ^n prescribes all players to refrain from investing with positive probability at time t. Let $\underline{s}_t^n = \inf\{\underline{s}_t^n(\sigma^n) \mid \sigma^n \in \Sigma^n\}$. Call $\underline{s}^n = (\underline{s}_1^n, \underline{s}_2^n)$ the strategy in which all players invest at time one if and only if $s_1^i > \underline{s}_1^n$ and in which all active players invest at time two if and only if $s_2^i > \underline{s}_2^n$. Let

$$\hat{g}(s_1^i, \hat{\sigma}^n, \sigma^n) \equiv (s_1^i - 1)(1 - \frac{\tau}{2}(I_{\{h(s_1^i-\epsilon,\sigma^n)>0\}} + I_{\{h(s_1^i+\epsilon,\sigma^n)>0\}}))$$

$$+\frac{1}{2}E_2(n_1|\hat{\sigma}^n, s_1^i - \epsilon)(1 - \tau I_{\{h(s_1^i-\epsilon,\sigma^n)>0\}}) + \frac{1}{2}E_2(n_1|\hat{\sigma}^n, s_1^i + \epsilon)(1 - \tau I_{\{h(s_1^i+\epsilon,\sigma^n)>0\}})$$

$$+\frac{1}{2}E_2(n_2|\hat{\sigma}^n, s_1^i - \epsilon)(\alpha - \tau I_{\{h(s_1^i-\epsilon,\sigma^n)>0\}}) + \frac{1}{2}E_2(n_2|\hat{\sigma}^n, s_1^i + \epsilon)(\alpha - \tau I_{\{h(s_1^i+\epsilon,\sigma^n)>0\}})$$

$$-\frac{\tau}{2}(I_{\{h(s_1^i-\epsilon,\sigma^n)>0\}}(-\epsilon - \Delta) + I_{\{h(s_1^i+\epsilon,\sigma^n)>0\}}(\epsilon - \Delta)),$$

where $I_{\{.\}}$ denotes the indicator function, and $\hat{\sigma}^n \in \Sigma^n$. In words, $\hat{g}(s_1^i, \hat{\sigma}^n, \sigma^n)$ denotes the difference between player i's gain of investing and her gain of waiting given that all the other players follow strategy $\hat{\sigma}^n$ and given that, at time two, player i decides to invest or not, *under the assumption that all players follow strategy* σ^n *instead of* $\hat{\sigma}^n$. Trivially, $\hat{g}(s_1^i, \sigma^n, \sigma^n) = g(s_1^i, \sigma^n)$. We first state and prove the following lemma.

LEMMA 5 *If* $\alpha \geq \tau$, $g(s_1^i, \underline{s}^n) \geq g(s_1^i, \sigma^n)$ $\forall s_1^i$ *and* $\forall \sigma^n \in \Sigma^n$.

Proof: Observe that

$$g(s_1^i, \underline{s}^n) - \hat{g}(s_1^i, \sigma^n, \underline{s}^n) = \frac{1}{2} \sum_{s_2^i \in \{s_1^i-\epsilon, s_1^i+\epsilon\}} \{[E_2(n_1|\underline{s}^n, s_2^i) \tag{7}$$

$$-E_2(n_1|\sigma^n, s_2^i)](1-\tau I_{\{h(s_2^i,\underline{s}^n)>0\}})+[E_2(n_2|\underline{s}^n, s_2^i)-E_2(n_2|\sigma^n, s_2^i)](\alpha-\tau I_{\{h(s_2^i,\underline{s}^n)>0\}})\}.$$

For each s_2^i, define the expression between {...} of the above equation as

$$f(s_2^i, \sigma^n) \equiv (1 - \tau I_{\{h(s_2^i,\underline{s}^n)>0\}})m(s_2^i, \sigma^n) + (\alpha - \tau I_{\{h(s_2^i,\underline{s}^n)>0\}})m'(s_2^i, \sigma^n),$$

where $m(s_2^i, \sigma^n)$ and $m'(s_2^i, \sigma^n)$ are defined as

$$m(s_2^i, \sigma^n) \equiv \frac{1}{2\epsilon} \int_{s_2^i - \epsilon}^{s_2^i + \epsilon} (n_1(\theta, \underline{s}^n) - n_1(\theta, \sigma^n)) d\theta,$$

$$m'(s_2^i, \sigma^n) \equiv \frac{1}{2\epsilon} \int_{s_2^i - \epsilon}^{s_2^i + \epsilon} (n_2(\theta, \underline{s}^n) - n_2(\theta, \sigma^n)) d\theta.$$

As all players with a $s_1^i > \underline{s}_1^n$ invest under $\underline{s}^n$, it follows that $m(s_2^i, \sigma^n) \geq 0$ for all s_2^i and for all σ^n. Hence, if $m'(s_2^i, \sigma^n) \geq 0$, then $f(s_2^i, \sigma^n)$ is positive. Thus suppose that $m'(s_2^i, \sigma^n) < 0$. Then $f(s_2^i, \sigma^n)$ is bounded below by $(1 - \tau I_{\{h(s_2^i, \underline{s}^n) > 0\}})(m(s_2^i, (\sigma_1^n, \underline{s}_2^n)) + m'(s_2^i, (\sigma_1^n, \underline{s}_2^n)))$. Note that

$$m(s_2^i, (\sigma_1^n, \underline{s}_2^n)) + m'(s_2^i, (\sigma_1^n, \underline{s}_2^n)) =$$

$$\frac{1}{2\epsilon} \int_{s_2^i - \epsilon}^{s_2^i + \epsilon} [(n_1(\theta, \underline{s}^n) + n_2(\theta, \underline{s}^n)) - (n_1(\theta, (\sigma_1^n, \underline{s}_2^n)) + n_2(\theta, (\sigma_1^n, \underline{s}_2^n)))] d\theta,$$

which is nonnegative and we conclude that $g(s_1^i, \underline{s}^n) \geq \hat{g}(s_1^i, \sigma^n, \underline{s}^n)$. As $g(s_1^i, \sigma^n)$ prescribes optimal time-two behavior, trivially $\hat{g}(s_1^i, \sigma^n, \underline{s}^n) \geq g(s_1^i, \sigma^n)$, and thus $g(s_1^i, \underline{s}^n) \geq g(s_1^i, \sigma^n)$ $\forall \sigma^n \in \Sigma^n$. Q.E.D.

Let $\overline{s}_t^n(\sigma^n)$ be the infimum above which σ^n prescribes all players to invest at time t with probability 1. Let $\overline{s}_t^n = \sup\{\overline{s}_t^n(\sigma^n) \mid \sigma^n \in \Sigma^n\}$. Call $\overline{s}^n = (\overline{s}_1^n, \overline{s}_2^n)$ the strategy in which all players invest at time one if and only if $s_1^i > \overline{s}_1^n$ and in which all active players invest at time two if and only if $s_2^i > \overline{s}_2^n$.

Proof of Proposition 2:

It follows from Lemma (5) that, conditional on s_1^i, investing in the first period is dominated after $n+1$ rounds of iterative elimination of dominated strategies if and only if $g(s_1^i, \underline{s}^n) < 0$.

Furthermore, $(n_1 + n_2)(\theta, \underline{s}^n) \geq (n_1 + n_2)(\theta, \sigma^n)$ for all $\sigma^n \in \Sigma^n$ because $\underline{s}^n$ prescribes an active player to invest whenever investing is not dominated after n rounds of iterative elimination of dominated strategies. Using this fact, it is easy to check that $h(s_2^i, \underline{s}^n) \geq h(s_2^i, \sigma^n)$ for all $\sigma^n \in \Sigma^n$. Hence, conditional on s_2^i, investing in the second period is dominated after $n+1$ rounds of iterative elimination if and only if $h(s_2^i, \underline{s}^n) < 0$. We conclude that when iteratively deleting dominated strategies from below, we can restrict attention to switching strategies $\underline{s}^n$.

We now show by induction that $\underline{s}_1^n$ and $\underline{s}_2^n$ are increasing sequences. Trivially, $(\underline{s}_1^0, \underline{s}_2^0) = (-\infty, -\infty)$. Because it is a dominant strategy not to invest for sufficiently low first- and second-period signals, $(\underline{s}_1^1, \underline{s}_2^1) >> (\underline{s}_1^0, \underline{s}_2^0)$. We are left to show that, $\underline{s}^{n-2} \leq \underline{s}^{n-1}$, implies that $\underline{s}^{n-1} \leq \underline{s}^n$.[19]

We first show that $\underline{s}_1^{n-1} \leq \underline{s}_1^n$. By definition of $\underline{s}_1^{n-1}$, we know that $g(\underline{s}_1^{n-1}, \underline{s}^{n-2}) = 0$. As $\underline{s}^{n-1} \in \Sigma^{n-2}$, from Lemma (5) we know that $g(\underline{s}_1^{n-1}, \underline{s}^{n-2}) \geq g(\underline{s}_1^{n-1}, \underline{s}^{n-1})$. $\underline{s}_1^n$ cannot be strictly lower than $\underline{s}_1^{n-1}$ because, by definition of $\underline{s}_1^{n-1}$, $\forall s_1^i < \underline{s}_1^{n-1}$ all strategies which prescribe player i to invest at time one with positive probability are dominated ones. As $g(\underline{s}_1^{n-1}, \underline{s}^{n-1}) \leq 0$ and as there exists some $\tilde{s}_1^i$ for which it is a dominant strategy to invest (i.e. for which $g(\tilde{s}_1^i, \underline{s}^{n-1}) > 0$), continuity of $g(\cdot)$ implies that there exists $\underline{s}_1^n$ such that $g(\underline{s}_1^n, \underline{s}^{n-1}) = 0$.

Next, we show that $\underline{s}_2^{n-1} \leq \underline{s}_2^n$. It is obvious that $E_2(n_1 + n_2 | s_2^i, \underline{s}^{n-2}) \geq E_2(n_1 + n_2 | s_2^i, \underline{s}^{n-1})$. Therefore,

$$\underline{s}_2^{n-1} + E_2(n_1 + n_2 | \underline{s}_2^{n-1}, \underline{s}^{n-1}) - 1 - \Delta \leq \underline{s}_2^{n-1} + E_2(n_1 + n_2 | \underline{s}_2^{n-1}, \underline{s}^{n-2}) - 1 - \Delta = 0$$

This implies that $\underline{s}_2^{n-1} \leq \underline{s}_2^n$ because $s_2^i + E_2(n_1 + n_2 | s_2^i, \underline{s}^{n-1}) - 1 - \Delta$ is strictly increasing in s_2^i and by definition

$$\underline{s}_2^n + E_2(n_1 + n_2 | \underline{s}_2^n, \underline{s}^{n-1}) - 1 - \Delta = 0.$$

Hence, as $n \to \infty$, $\underline{s}^n$ converges to some cutoff vector $\underline{s}$ that satisfies $g(\underline{s}_1, \underline{s}) = 0$ and $h(\underline{s}_2, \underline{s}) = 0$. Using reasoning that mirrors the one for $\underline{s}^n$, shows that $\overline{s}^n$ is a decreasing sequence that, as $n \to \infty$, converges to a cutoff vector $\overline{s}$ that satisfies $g(\overline{s}_1, \overline{s}) = 0$ and $h(\overline{s}_2, \overline{s}) = 0$. Observe that, by construction of $\overline{s}$ and $\underline{s}$, $\underline{s}_1 \leq \overline{s}_1$ and $\underline{s}_2 \leq \overline{s}_2$.

We are left to show that $\underline{s} = \overline{s}$. Suppose otherwise. Both $\underline{s}$ and $\overline{s}$ solve the following system of equations.

$$g(k_1, k) = 0, \tag{8}$$

$$h(k_2, k) = 0. \tag{9}$$

First, observe that if $\underline{s}_1 = \overline{s}_1$ then $\underline{s}_2 = \overline{s}_2$ because $h(k_2, (k_1, k_2))$ is strictly increasing in k_2. Thus, $\underline{s}_1 < \overline{s}_1$. $\underline{s}_2$ and $\overline{s}_2$ must be chosen such that $h(\underline{s}_2, \underline{s})$ and

[19] When comparing two vectors, we use $\leq$ to indicate that for all i, the ith component of the first vector is $\leq$ to the ith component of the second vector.

$h(\overline{s}_2, \overline{s}) = 0$, which implies that

$$(10) \qquad (\overline{s}_2 - \underline{s}_2) = E_2(n_1 + n_2 | \underline{s}_2, \underline{s}) - E_2(n_1 + n_2 | \overline{s}_2, \overline{s})$$

To gain some insight about $E_2(n_1+n_2|\cdot)$'s consider the following two pair of cutoffs (k_1', k_2) and (k_1'', k_2). Both strategies possess the same second-period cutoffs but suppose without loss of generality that $k_1' < k_1''$. Consider any arbitrary (s_1^i, s_2^i). Clearly, if player i invests in either period under strategy (k_1'', k_2), she also invests under strategy (k_1', k_2). Hence, $E_2(n_1 + n_2 | k_2, (k_1, k_2))$ is weakly increasing in $k_2 - k_1$. Hence a necessary condition for for equation (10) to hold is that

$$(11) \qquad \overline{s}_2 - \overline{s}_1 < \underline{s}_2 - \underline{s}_1.$$

Now consider a player whose $s_2^i = k_1 + \epsilon$. Consider two different second-period cut-off levels k_2' and k_2'', and suppose without loss of generality that $k_2' < k_2''$. Consider any $s_1^i < k_1$. Clearly, if player i invests at time two under (k_1, k_2''), she will also do so under (k_1, k_2'). Hence, $E_2(n_2 | k_1 + \epsilon, (k_1, k_2))$ is weakly decreasing in $k_2 - k_1$. The same logic can be applied to $E_2(n_2 | k_1 - \epsilon, (k_1, k_2))$. Thus from (11) follows that

$$(12) \quad E_2(n_2|\overline{s}_1 + \epsilon, \overline{s}) \geq E_2(n_2|\underline{s}_1 + \epsilon, \underline{s}) \text{and} E_2(n_2|\overline{s}_1 - \epsilon, \overline{s}) \geq E_2(n_2|\underline{s}_1 - \epsilon, \underline{s}).$$

We also know that $\underline{s}_1$ and $\overline{s}_1$ must be chosen such that $g(\underline{s}_1, \underline{s}) = g(\overline{s}_1, \overline{s}) = 0$. Note that

$$\begin{aligned} g(\overline{s}_1, \overline{s}) - \hat{g}(\underline{s}_1, \underline{s}, \overline{s}) &= (\overline{s}_1 - \underline{s}_1)(1 - \frac{\tau}{2} I_{\{h(\overline{s}_1+\epsilon, \overline{s})>0\}} - \frac{\tau}{2} I_{\{h(\overline{s}_1-\epsilon, \overline{s})>0\}}) \\ &+ \frac{1}{2}(E_2(n_2|\overline{s}_1 + \epsilon, \overline{s}) - E_2(n_2|\underline{s}_1 + \epsilon, \underline{s}))(\alpha - \tau I_{\{h(\overline{s}_1+\epsilon, \overline{s})>0\}}) \\ &+ \frac{1}{2}(E_2(n_2|\overline{s}_1 - \epsilon, \overline{s}) - E_2(n_2|\underline{s}_1 - \epsilon, \underline{s}))(\alpha - \tau I_{\{h(\overline{s}_1-\epsilon, \overline{s})>0\}}). \end{aligned}$$

From (12) and from the fact that $\overline{s}_1 > \underline{s}_1$, it follows that $g(\overline{s}_1, \overline{s}) > \hat{g}(\underline{s}_1, \underline{s}, \overline{s})$. As $\hat{g}(\underline{s}_1, \underline{s}, \overline{s}) \geq g(\underline{s}_1, \underline{s})$ it follows that, under (11), $g(\overline{s}_1, \overline{s}) > g(\underline{s}_1, \underline{s})$, a contradiction. Q.E.D.

Proof of Proposition 3:

In an immediate investment equilibrium no player invests in the second period. Hence,

$$h(s_2^i, k) = s_2^i + \gamma E(n_1 | s_2^i, k) - 1 - \Delta.$$

It follows from the derivation of $n_1(\theta, k)$ in Lemma 3 that $n_1(\theta, k)$ is weakly increasing in the fundamental θ. Thus,

$$E(n_1|s_2^i, k) = \frac{1}{2\epsilon} \int_{s_2^i-\epsilon}^{s_2^i+\epsilon} n_1(\theta, k)\ d\theta$$

is weakly increasing in s_2^i, and hence $h(s_2^i, k)$ is strictly increasing in an immediate investment equilibrium. Therefore, there exists a unique k_2 such that $h(s_2^i, k) \leq 0$ if and only if $s_2^i \leq k_2$. By definition, we look for an equilibrium in which $k_2 \geq k_1+\epsilon$, which implies that $h(s_2^i = k_1+\epsilon, k) \leq 0$. Hence, the gain of waiting must be equal to zero. Therefore, k_1 must be set such that a player who possesses a signal $s_1^i = k_1$ is indifferent between investing and not investing. Thus k_1 solves the following equation

$$k_1 + E(n_1 \mid s_1^i = k_1, k) - 1 = 0.$$

Using the function $n_1(\theta, k)$, derived in Lemma 3, and the fact that

$$E(n_1 \mid s_1^i = k_1, k) = \frac{1}{4\epsilon} \int_{k_1-2\epsilon}^{k_1+2\epsilon} n_1(\theta, k)\ d\theta,$$

it is easy to verify that $E(n_1 \mid s_1^i = k_1, k) = \frac{1}{2}$. Thus, in an immediate investment equilibrium $k_1 = \frac{1}{2}$. Using this fact to rewrite the condition that no player has an incentive to invest in the second period, i.e. that $h(s_2^i = k_1 + \epsilon, k) \leq 0$, gives

$$\frac{1}{2} + \epsilon + \gamma E(n_1 \mid s_2^i = k_1 + \epsilon, k) \leq 1 + \Delta. \tag{13}$$

Similarly, using $n_1(\theta, k)$ and the fact that

$$E(n_1 \mid s_2^i = k_1 + \epsilon, k) = \frac{1}{2\epsilon} \int_{k_1}^{k_1+2\epsilon} n_1(\theta, k)\ d\theta,$$

it is easy to verify that $E(n_1 \mid s_2^i = k_1 + \epsilon, k) = \frac{3}{4}$. Substituting this into equation (13) and rewriting yields $\Delta \geq -\frac{1}{2} + \epsilon + \frac{3}{4}\gamma$, which is a necessary condition for an immediate investment equilibrium to exist. Because we already established that $h(s_2^i, k)$ is strictly increasing, it suffices to show that $g(\cdot)$ is (weakly) increasing to show that an immediate investment equilibrium exists whenever $\Delta \geq -\frac{1}{2} + \epsilon + \frac{3}{4}\gamma$. First, observe that for all $s_1^i < k_2 - \epsilon$, one has

$$g(s_1^i, k) = s_1^i + E(n_1 \mid s_1^i, k) - 1,$$

which is strictly increasing in s_1^i because $E(n_1 \mid s_1^i, k)$ is weakly increasing in s_1^i. Second, for all $k_2 - \epsilon < s_1^i < k_2 + \epsilon$,

$$g(s_1^i, k) = s_1^i + E(n_1 \mid s_1^i, k) - 1 - \frac{\tau}{2} h(s_1^i + \epsilon, k).$$

Using Lemma (2) and equation (1), one can rewrite the above equation as

$$g(s_1^i, k) = (1-\frac{\tau}{2})s_1^i + \frac{1}{2}[E(n_1 \mid s_2^i = s_1^i - \epsilon, k) + (1-\tau\gamma)E(n_1 \mid s_2^i = s_1^i + \epsilon, k) - \tau\epsilon - (2-\tau) + \tau\Delta].$$

Since $E(n_1 \mid s_2^i = s_1^i - \epsilon, k)$ and $E(n_1 \mid s_2^i = s_1^i + \epsilon, k)$ are weakly increasing in s_1^i, and $\tau, \gamma \leq 1$, $g(s_1^i, k)$ is strictly increasing in s_1^i in this subcase. Third, for all $k_2 + \epsilon < s_1^i$, one has

$$g(s_1^i, k) = s_1^i + E(n_1 \mid s_1^i, k) - 1 - \frac{\tau}{2}[h(s_1^i - \epsilon, k) + h(s_1^i + \epsilon, k)].$$

Rewriting this equation using Lemma (2) and equation (1) yields

$$g(s_1^i, k) = (1-\tau)s_1^i + \frac{(1-\tau\gamma)}{2}[E(n_1 \mid s_2^i = s_1^i - \epsilon) + E(n_1 \mid s_2^i = s_1^i + \epsilon)] - (1-\tau) + \tau\Delta.$$

Since $E(n_1 \mid s_2^i = s_1^i - \epsilon)$ and $E(n_1 \mid s_2^i = s_1^i + \epsilon)$ are weakly increasing in s_1^i, and $\tau, \gamma \leq 1$, $g(s_1^i, k)$ is weakly increasing in s_1^i in this subcase. Q.E.D.

Proof of Proposition 4:

Rewriting (4) and (5) using the fact that $k_1 - \epsilon < k_2 < k_1 + \epsilon$ in an informative waiting equilibrium gives

$$k_1 + \frac{1}{2} + (\frac{\alpha}{8\epsilon})(k_1 + \epsilon - k_2) - 1 - \frac{\tau}{2}\{k_1 + \epsilon + \frac{3\gamma}{4} + \frac{1}{16\epsilon^2}(k_1 + \epsilon - k_2)(k_2 + 3\epsilon - k_1) - 1 - \Delta\} = 0,$$

$$k_2 + \gamma\{\frac{1}{4} + \frac{1}{4\epsilon}(k_2 + \epsilon - k_1) + \frac{1}{16\epsilon^2}(k_1 + \epsilon - k_2)(k_2 + 3\epsilon - k_1)\} - 1 - \Delta = 0.$$

Thus, (4) and (5) are a pair of quadratic equations, which is equivalent to a fourth order polynomial. Hence, there exists a routine procedure to solve this system of equations. Using mathematica to solve this system of equations shows that there are only two pair of roots (k_{11}, k_{21}) and (k_{21}, k_{22}). Rewriting, gives the expressions given in the proposition above. Because (4) and (5) are necessary conditions for an equilibrium, all informative waiting equilibria ar either of the form (k_{11}, k_{21}) or (k_{21}, k_{22}).

Observe that all roots are real if and only if $D \geq 0$. This requires that

$$16\epsilon - 8 + 12\gamma + [(2-\alpha) - (2-\tau)x]^2 \geq 16\Delta.$$

Rewriting gives condition (a).

(k_{11}, k_{21}) is a valid solution only if $k_{11} - \epsilon < k_{21} < k_{11} + \epsilon$, because otherwise the functional form of (4) and (5) would differ from the one used above. That is, we require that (i) $-\epsilon < k_{11} - k_{21}$ and that (ii) $k_{11} - k_{21} < \epsilon$. Using the fact that

$$k_{11} - k_{21} = \epsilon[1 - \alpha - (2 - \tau)x + \sqrt{D}], \tag{14}$$

condition (i) holds if and only if

$$(2 - \tau)x - (2 - \alpha) < \sqrt{D}.$$

Note that this inequality is satisfied if either $(2 - \alpha) > (2 - \tau)x$ or if

$$[(2 - \tau)x - (2 - \alpha)]^2 < -16\Delta + 16\epsilon - 8 + 12\gamma + [(2 - \alpha) - (2 - \tau)x]^2.$$

Rewriting gives condition (b).

Using $k_{11} - k_{21} = \epsilon[1 - \alpha - (2 - \tau)x + \sqrt{D}]$, to rewrite condition (ii) gives

$$\sqrt{D} < \alpha + (2 - \tau)x.$$

Squaring this inequality on both sides and rewriting yields

$$-16\Delta + 16\epsilon + 12\gamma - 4(1 + \alpha) - 4x(2 - \tau) < 0,$$

which is equivalent to condition (c) in the proposition. Hence, conditions (a), (b), and (c) are necessary conditions for (k_{11}, k_{21}) to characterize an equilibrium.

Similarly, (k_{12}, k_{22}) is a valid solution only if both (i) $-\epsilon < k_{12} - k_{22}$ and (ii) $k_{12} - k_{22} < \epsilon$ hold. Using the fact that $k_{12} - k_{22} = \epsilon[1 - \alpha - (2 - \tau)x - \sqrt{D}]$, condition (i) holds if and only if $\sqrt{D} < (2 - \alpha) - (2 - \tau)x$. Hence, condition (i) requires that $(2-\alpha) > (2-\tau)x$, which is condition (d) in the proposition, and that $D < [(2-\alpha)-(2-\tau)x]^2$, which is equivalent to condition (e) in the proposition. We conclude that conditions (a), (d) and (e) are necessary conditions for (k_{12}, k_{22}) to characterize an equilibrium. (Note also that $k_{12}-k_{22} = \epsilon[1-\alpha-(2-\tau)x-\sqrt{D}] < \epsilon$.) Hence, we have established that no other informative waiting equilibrium than the ones characterized in the proposition exist. To show that (k_{11}, k_{21}) and (k_{12}, k_{22}) are indeed equilibria under the above conditions, we are left to verify that (i) $h(s_2^i, k) < 0$ for all $s_2^i < k_2$, (ii) $h(s_2^i, k) > 0$ $\forall s_2^i \in (k_2, k_1 + \epsilon]$, and that (iii) $g(s_1^i, k) < 0$ if and only if $s_1^i < k_1$. Conditions (i) and (ii) follow from Lemma (4). The proof of Condition (iii) is available upon request. Q.E.D.

Proof of Lemma 1:

We start by proving the first statement. Suppose otherwise, i.e. there exists a symmetric switching equilibrium in which no player invests in the first period. Rewriting condition (5) gives

$$h(s_2^i, k) = k_2 + E(n_2 \mid s_2^i = k_2, k) - 1 - \Delta = 0.$$

It is easy to check that $E(n_2 \mid s_2^i = k_2, k) = \frac{1}{2}$, and hence $k_2 = \frac{1}{2} + \Delta$. Consider a player with a signal $s_1^i > k_2 + 3\epsilon$. This player knows in equilibrium that all (other) players invest in the second period. Hence,

$$g(s_1^i, k) = s_1^i + \alpha - 1 - \tau[s_1^i - \Delta].$$

Rewriting, yields $g(s_1^i, k) = (1-\tau)s_1^i - (1-\alpha) + \tau\Delta$. If $\tau < 1$, then for all sufficiently high s_1^i one has $g(s_1^i, k) > 0$. Similarly if $\alpha = \gamma = \tau = 1$ and $\Delta > 0$, then $g(s_1^i, k) = \Delta > 0$. But if $g(s_1^i, k) > 0$ a player has a strict incentive to invest in the first period, which contradicts the fact that all players refrain from investing in the first period.
We now prove the second statement. Recall that $E(n_1 \mid s_1^i = k_1, k) = \frac{1}{2}$. Also, note that since $k_2 < k_1 - \epsilon$, a player with signal $s_1^i = k_1$ who waits will invest in the second period for certain. Using these facts and Lemma (2) to rewrite equilibrium conditions (4) and (5) gives:

$$k_1(1-\tau) + \frac{1}{2}(1-\tau\gamma) + (\alpha-\tau)E(n_2 \mid s_1^i = k_1, k) - 1 + \tau(1+\Delta) = 0, \tag{15}$$

$$k_2 = 1 + \Delta - \gamma E(n_1 \mid s_2^i = k_2, k) - E(n_1 \mid s_2^i = k_2, k). \tag{16}$$

Observe that any player with a signal $s_2^i < k_2$ does not invest in either period because in this case $s_1^i \leq s_2^i + \epsilon < k_2 + \epsilon < k_1$. Hence, $E(n_1 + n_2 \mid s_2^i = k_2, k) = \frac{1}{2}$ and thus $E(\gamma n_1 + n_2 \mid s_2^i = k_2, k_2) \leq \frac{1}{2}$. Using this fact and equation (16), we conclude that $k_2 \geq \frac{1}{2} + \Delta$.

Rewriting equation (15) shows that

$$k_1 = \frac{1}{1-\tau}\{\frac{1}{2}(1+\tau\gamma - \tau(1+\Delta) + (\tau-\alpha)E(n_2 \mid s_1^i = k_1, k))\}.$$

First, suppose that $\alpha \geq \tau$. In this case

$$k_1 \leq \frac{1}{1-\tau}\{\frac{1}{2}(1+\tau\gamma - \tau(1+\Delta)\}$$

and since $k_2 \geq \frac{1}{2} + \Delta$ a necessary condition for $k_1 > k_2 + \epsilon$ is that

$$\{\frac{1}{2}(1 + \tau\gamma) - \tau(1 + \Delta)\} \geq (1 - \tau)(\frac{1}{2} + \Delta + \epsilon).$$

This is equivalent to $0 \geq \Delta + \epsilon(1 - \tau) + \frac{\tau}{2}(1 - \gamma)$, a contradiction.

We are left to consider the case in which $\alpha < \tau$. Observe that since $E(n_1 \mid s_1^i = k_1, k) = \frac{1}{2}$ and $n_1 + n_2 \leq 1$, $E(n_2 \mid s_1^i = k_1, k) \leq \frac{1}{2}$. Hence,

$$k_1 \leq \frac{1}{1 - \tau}\{\frac{1}{2}(1 + \tau\gamma - \tau(1 + \Delta) + (\tau - \alpha)\frac{1}{2}\}.$$

Thus, a necessary condition for $k_1 > k_2 + \epsilon$ is that

$$\{\frac{1}{2}(1 + \tau\gamma - \tau(1 + \Delta) + (\tau - \alpha)\frac{1}{2}\} \geq (1 - \tau)(\frac{1}{2} + \Delta + \epsilon).$$

Rewriting this condition establishes the Lemma. Q.E.D.

The following Lemma is used to prove Proposition 6 and Proposition 7.

LEMMA 6 *Consider equilibria for which $k_2 \in (k_1 - \epsilon, k_1 + \epsilon)$. Then $E(n_2|k)$ increases if and only if $k_1 - k_2$ increases.*

Proof: Let $\kappa \equiv k_1 - \theta$. Rewrite $n_2(\theta, k)$ given in Lemma 3 as

$$n_2(\kappa, k_1 - k_2) = \begin{cases} 0 & \text{if } k_1 - k_2 + \epsilon \leq \kappa, \\ \frac{\epsilon - \kappa + (k_1 - k_2)}{4\epsilon} & \text{if } 0 \leq \kappa \leq k_1 - k_2 + \epsilon, \\ \frac{\epsilon + (k_1 - k_2)}{4\epsilon} & \text{if } k_1 - k_2 - \epsilon \leq \kappa \leq 0, \\ \frac{2\epsilon - \kappa}{4\epsilon} & \text{if } -2\epsilon \leq \kappa \leq k_1 - k_2 - \epsilon, \\ 0 & \text{if } \kappa \leq -2\epsilon. \end{cases}$$

Call $f(\kappa)$ the density function of κ. As θ is uniformly distributed over a large enough interval, $f(\kappa)$ is also uniformly distributed over the relevant interval. Hence,

$$E(n_2|k_1 - k_2) = f(\kappa)\int_{-2\epsilon}^{k_1 - k_2 + \epsilon} n_2(\kappa, k_1 - k_2)d\kappa.$$

Note that (i) $\forall\kappa, n_2(\kappa, k_1 - k_2)$ is (weakly) increasing in $k_1 - k_2$ and that (ii) the upper boundary of the integral also increases with $k_1 - k_2$. These observations imply that $E(n_2|k_1 - k_2)$ increases if and only if $k_1 - k_2$ increases. Q.E.D.

Proof of Proposition 6:

In the PI interpretation $\alpha = \tau = \gamma = 1$ and $\Delta > 0$. Thus, Proposition 2 implies that the equilibrium is unique. For all $\Delta \geq \Delta^b$, the unique equilibrium is the immediate investment equilibrium characterized in Proposition 3. In this equilibrium a player invests in the first period if and only if she receives a signal $s_1^i > 1/2$. For $\Delta < \Delta^b$ the unique equilibrium (k_{11}, k_{21}) is characterized in Proposition 4. Differentiating k_{11} and k_{21} with respect to Δ and rewriting, one has

$$\frac{\partial k_{11}}{\partial \Delta} = -\frac{4\epsilon}{\sqrt{D}},$$

$$\frac{\partial k_{21}}{\partial \Delta} = \frac{4\epsilon}{\sqrt{D}}.$$

The first part of the Proposition thus follows from the fact that as Δ approaches Δ^b, k_{11} approaches 1/2, which can be verified by substituting Δ^b into the expression for k_{11}.

$E(n_1 + n_2|k)$ is independent of Δ if

$$\frac{\partial E(n_1|k)}{\partial k_1}\frac{\partial k_1}{\partial \Delta} = -\frac{\partial E(n_2|k_1 - k_2)}{\partial (k_1 - k_2)}\frac{\partial (k_1 - k_2)}{\partial \Delta}. \tag{17}$$

Using Lemma 3, one has

$$E(n_1|k) = f(\theta)\int_{k_1-2\epsilon}^{k_1+2\epsilon} \frac{2\epsilon + \theta - k_1}{4\epsilon} d\theta + f(\theta)\int_{k_1+2\epsilon}^{\overline{\theta}} d\theta,$$

where the definition of $\overline{\theta}$ can be found in section 5. Integrating shows that $E(n_1|k) = f(\theta)[\overline{\theta} - k_1]$. Thus,

$$\frac{\partial E(n_1|k)}{\partial k_1} = -f(\theta).$$

Differentiating 14 yields

$$\frac{\partial (k_1 - k_2)}{\partial \Delta} = -\frac{4\epsilon}{\sqrt{D}} = 2\frac{\partial k_1}{\partial \Delta}.$$

Using Lemma 3, one has

$$E(n_2|k) = f(\theta)[\int_{k_2-\epsilon}^{k_1} \frac{\epsilon - k_2 + \theta}{4\epsilon} d\theta + \int_{k_1}^{k_2+\epsilon} \frac{k_1 - k_2 + \epsilon}{4\epsilon} d\theta + \int_{k_2+\epsilon}^{k_1+2\epsilon} \frac{k_1 + 2\epsilon - \theta}{4\epsilon} d\theta.$$

Integrating shows that $E(n_2|k) = \frac{f(\theta)}{2}[k_1 - k_2 + \epsilon]$. Hence,

$$\frac{\partial E(n_2|k)}{\partial(k_1 - k_2)} = \frac{f(\theta)}{2}.$$

Thus, condition 17 can be rewritten as

$$-f(\theta)\frac{\partial k_1}{\partial \Delta} = -\frac{f(\theta)}{2}2\frac{\partial k_1}{\partial \Delta},$$

which is obviously satisfied. Q.E.D.

Proof of Proposition 7:

In the NC interpretation

$$\alpha = \tau = \delta \in (0,1), \Delta = 0, \tag{18}$$

and $\theta = \tilde{\theta} - s + 1 - (1-\delta)F$. From Propositions 2, 3, and 4 we know that in the essentially unique equilibrium

$$k_1 = \frac{1}{8}\{\delta(\delta-2)(1+4\epsilon)^2 + 2(1+4\epsilon)[1-(1-\delta)^2] + (1-\delta)^2 + 3 \\ +2[(1+4\epsilon)\delta - \delta]\sqrt{1+4\epsilon+4\epsilon^2(2-\delta)}\}. \tag{19}$$

If F increases, θ decreases and, as k_1 is independent of F, less players become members at time one.

Lemma 6 states that $E(n_2|k)$ is a strictly increasing function of $k_1 - k_2$. Using (18) to rewrite (14), yields

$$k_1 - k_2 = \epsilon[2\sqrt{1+4\epsilon+4\epsilon^2(2-\delta)} - (1+4\epsilon(2-\delta))]. \tag{20}$$

Hence $E(n_2|k)$ is independent of F. As $E(n_1|k)$ decreases in F and as $E(n_2|k)$ remains constant, we also conclude that $E(n_1 + n_2|k)$ is decreasing in F.

We now analyze how $E(n_1|k)$ and $E(n_2|k)$ vary with δ. First-period membership is increasing in δ if and only if $\frac{\partial\theta}{\partial\delta} = F > \frac{\partial k_1}{\partial\delta}$. Differentiating (19) and rearranging terms yields

$$\frac{\partial k_1}{\partial \delta} = \epsilon[\frac{1+4\epsilon+4\epsilon^2(2-\delta) - 2\epsilon^2\delta - 4\epsilon(1-\delta)\sqrt{1+4\epsilon+4\epsilon^2(2-\delta)}}{\sqrt{1+4\epsilon+4\epsilon^2(2-\delta)}}],$$

which goes to zero as $\epsilon \to 0$. Hence for a given $F > 0$ we can choose ϵ sufficiently small to ensure that $F > \frac{\partial k_1}{\partial \delta}$ and thus that first-period membership increases with δ. It is easy to check, however, that there also exist parameter values of ϵ, δ, F, for which $F < \frac{\partial k_1}{\partial \delta}$ in which case first period membership decreases in δ.

Differentiating (20) with respect to δ and rewriting yields

$$\frac{\partial(k_1 - k_2)}{\partial \delta} = 4\epsilon^2 [\frac{\sqrt{1 + 4\epsilon + 4\epsilon^2(2-\delta)} - \epsilon}{\sqrt{1 + 4\epsilon + 4\epsilon^2(2-\delta)}}],$$

which is strictly positive for all $\delta \leq 1$. This proves that $E(n_2|k)$ increases with δ. As $\epsilon \to 0$, $\frac{\partial(k_1-k_2)}{\partial \delta} \to 0$, and hence the increase in $E(n_2|k)$ goes to zero. Q.E.D.

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Figure One: Period-one distribution of signals (not to scale)

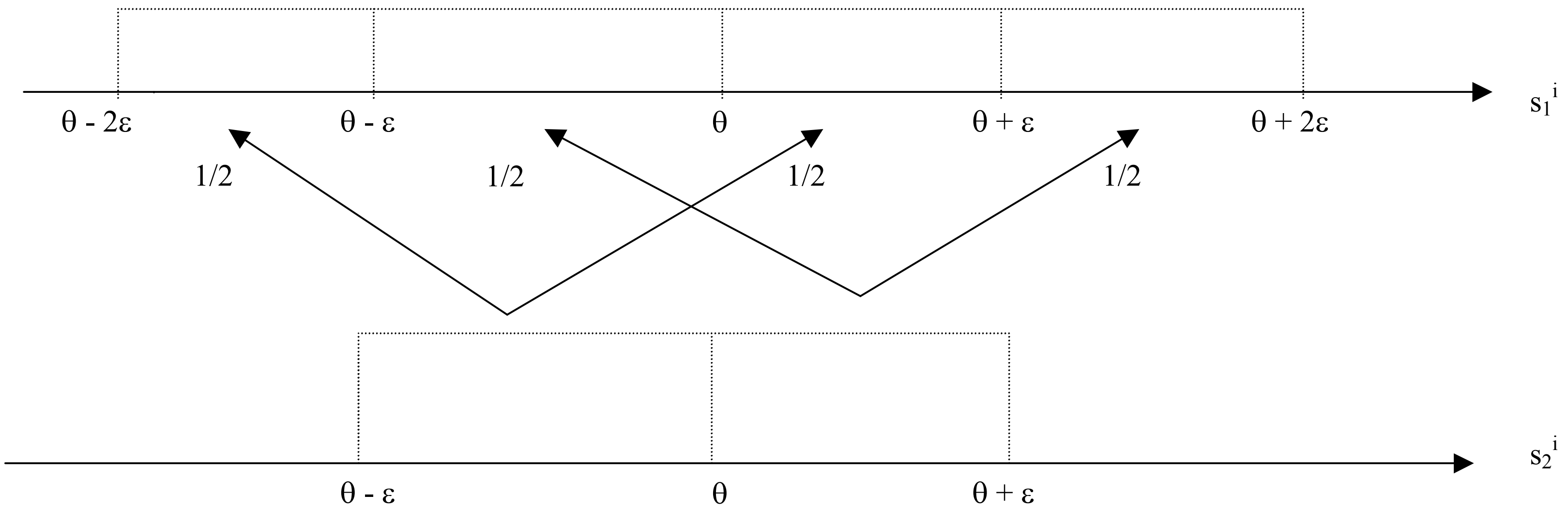

Figure 2: Graphical representation of the mass of players investing in the two periods as a function of (k_1,k_2).

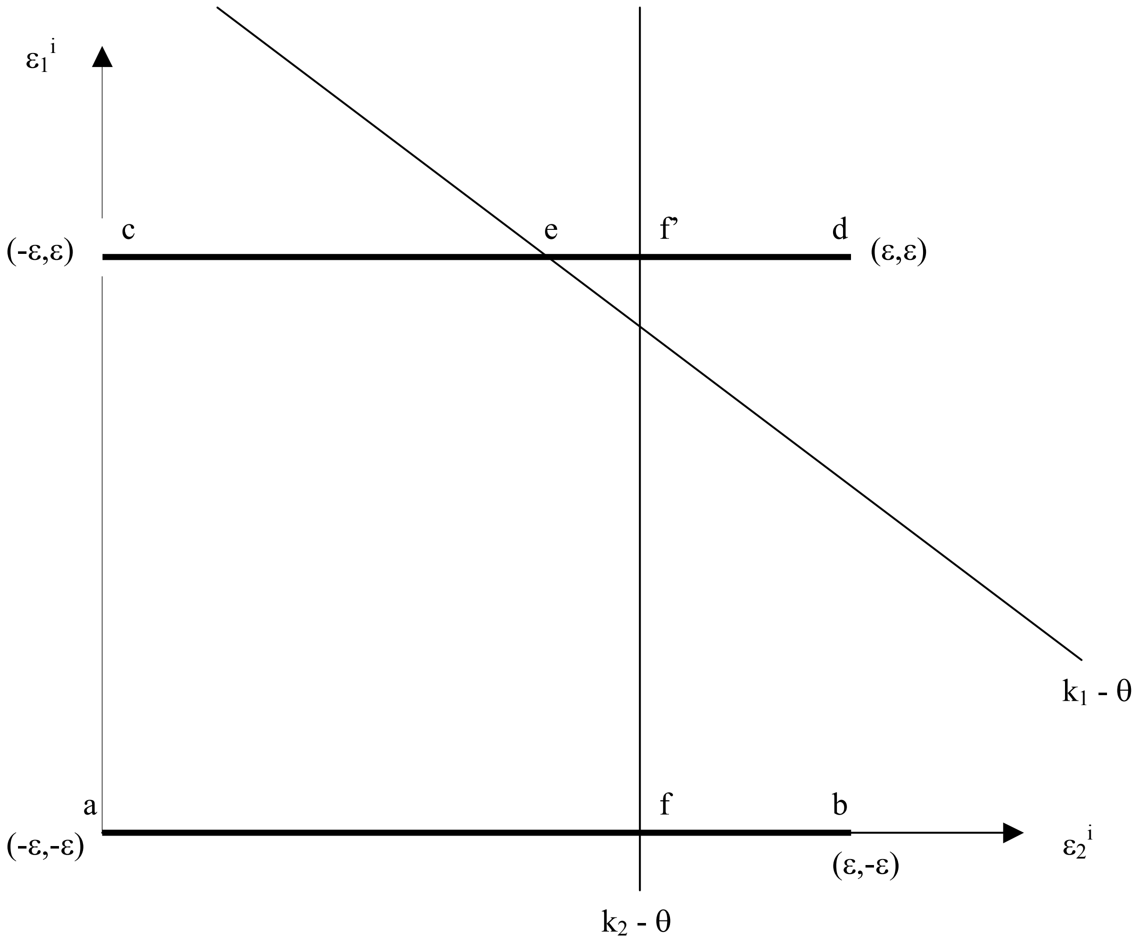

Figure 3: Graphical representation of the mass of players investing in the two periods as a function of (k_1,k_2).

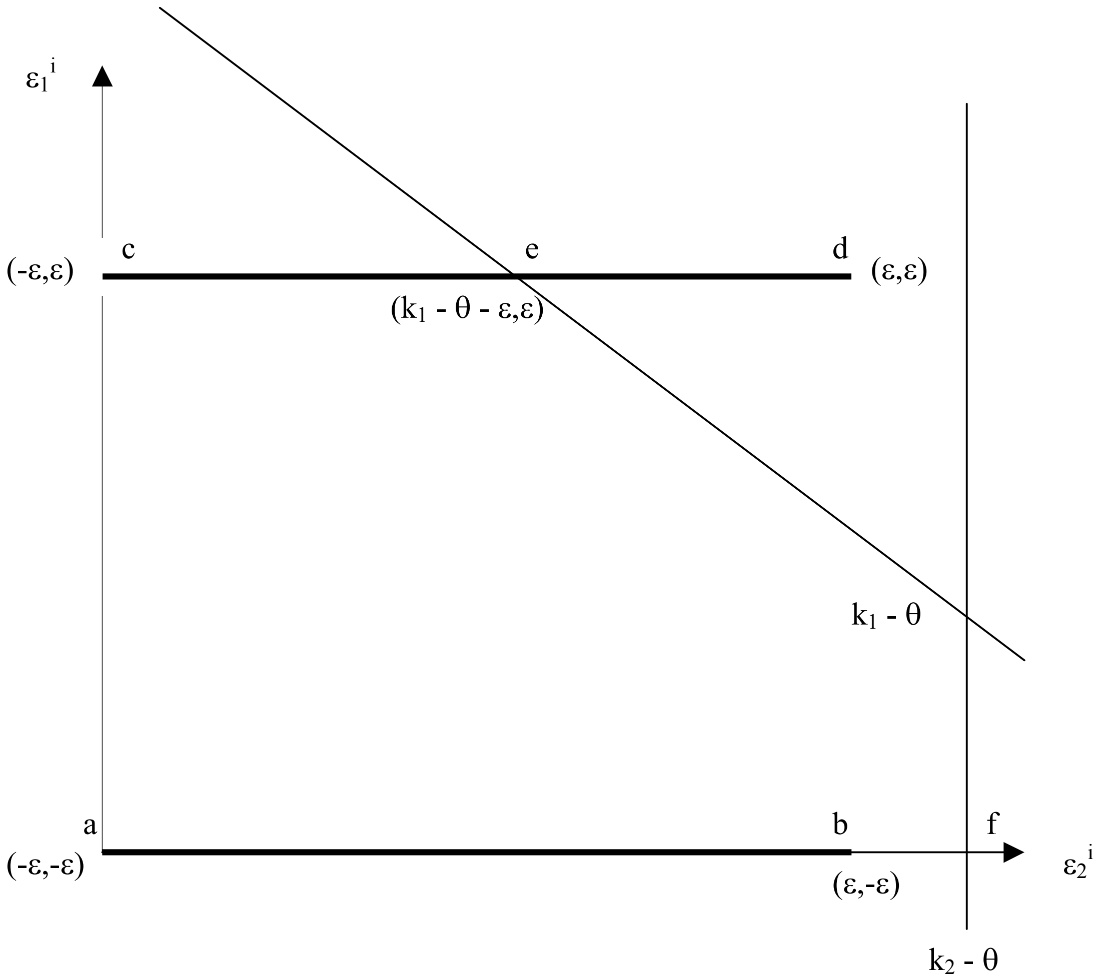

Figure 4: Graphical representation of the mass of players investing in the two periods as a function of (k_1,k_2).