Rational Exuberance*

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Abstract

We study a two-player investment game with information externalities. Necessary and sufficient conditions for a unique switching equilibrium are provided. When public news indicates that the investment opportunity is very profitable, too many types are investing early and investments should therefore be taxed. Conversely, any positive investment tax is suboptimally high if the public information is sufficiently unfavorable.

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1 Introduction

In 1996 the then chairman of the Fed Alan Greenspan famously criticized investors by warning that they were in the grip of "irrational exuberance". This catchy wording instantly made headlines and initially caused a sell-off of stock around the world. Greenspan, however, did not think that it was up to the Fed or to the federal government to intervene in the investment decisions of other agents. In particular, 30 months after his famous speech he came to the conclusion that:

> "How do you draw the line between a healthy, exciting economic boom and a . . . bubble . . . ? ... After thinking a great deal about this, I decided that ... the Fed would not second-guess "hundreds of thousands of informed investors." Instead the Fed would position itself to protect the economy in the event of a crash." (Greenspan, 2008, pages 200-1)

When investors are well-informed, Greenspan thus thought that policy makers should not interfere in their investment decisions—a rationale that we believe was shared broadly among policy makers in many countries. We question this rationale in a basic social-learning model and illustrate that even when investors are better informed than a policy maker, investments should be taxed when public news is sufficiently favorable.

Formally, we develop a two-player investment game to analyze this reasoning in more detail. There is a state of the world drawn from a normal distribution whose mean equals $\bar{\theta}$. If the state of the world is positive, both players should invest. If the state of the world is negative, no one should invest. Public news regarding the realization of the state of the world is captured by the prior mean $\bar{\theta}$. Public news can be favorable for different reasons. For example, there are favorable "stories" in the public domain about the realization of the fundamental or—perhaps as a consequence of past favorable public news—an investment boom occurred in the previous period. The public news is on average correct, but can sometimes be completely wrong. In particular, it is possible that the public news depicts the investment opportunity as a "golden" one, when in reality the state of the world is negative. To capture the idea that individual investors are better informed than the policy maker, we assume that both players get an additional normally distributed private signal about the realization of the state. Players combine the public news and their private information to compute the expected returns from investing. They then simultaneously decide whether or not to invest in period one. If a player invests, her payoff equals the state of the world. A player who has not invested in period one, observes the other player's period-one decision and then reconsiders her choice in period two. Payoffs from acting late are discounted, and a player who doesn't invest receives her outside option.

A rational player, by delaying her investment decision, can thus learn by observing the other player's first-period investment decision. The more optimistic a player, the less willing she is to delay as returns from investing at a later point in time are discounted. We therefore look for equilibria that are characterized by a first-period cutoff belief, above which a player invests immediately and below which she waits. Crucially, the value of waiting in our social-learning model depends on the other player's behavior. Whenever the other player's cutoff is sufficiently low, seeing her investing comes as no surprise. In this case an investment decision contains little information, which makes waiting for further information relatively undesirable. When the other player's cutoff becomes higher, she will invest less often. An investment decision then reveals that she has good private information which, in turn, makes waiting more desirable. Whenever this force is strong enough, multiple symmetric switching equilibria exist. Section 4 therefore investigates under what conditions the symmetric switching equilibrium is unique in a laissez-faire economy. We find among other conditions that if players are sufficiently patient, equilibrium is unique. Since it is often plausible in social-learning environments that players observe each other frequently, our model often predicts a unique symmetric switching equilibrium. The symmetric switching equilibrium is also unique whenever the public news is either sufficiently good or bad—i.e. if the public news suggests that the investment opportunity is a "golden" one, equilibrium is unique.

Building on this characterization, Sections 5 investigates the optimal investment cutoff that a social planner would want to implement. As mentioned above, we assume that the social planner knows the public news but has no additional private information regarding the investment opportunity.[1] We first establish that the social planner does not want to distort second-period investment cutoffs; in the final period rational players want to and should invest whenever the expected state of the world is positive. We also characterize the optimal first-period cutoff. In particular, we show that if the public news is sufficiently favorable, it is optimal to raise the first-period investment cutoff. Roughly speaking, if the public news is favorable, both players are very likely to invest early in the laissez-faire economy, which implies that the informational content of an investment decision is low. Raising the cutoff increases the informational content of the first-period investment decision and leads to a greater positive externality. Conversely, we also show that a social planner does not want to raise the cutoff if the public news is sufficiently unfavorable.

Section 6 establishes that the optimal cutoff can be implemented through a period-one investment tax (or subsidy). In particular, whenever the public news is sufficiently favorable, this is achieved through taxing first-period investment activity. The implementation, however, need not be unique even if the equilibrium of the laissez-faire economy is unique. Nevertheless, we establish that taxation is strictly optimal when the public news is sufficiently favorable by showing that there exist positive tax rates for which both equilibrium remains unique and welfare is higher.

Section 2 outlines the existing social-learning literature. We highlight that one needs to extend the canonical social-learning model to address our policy question: The standard model with a binary state and signal space, for example, is characterized by multiple equilibria and the optimality of taxing or subsidizing investments in this model depends on—in our view ad-hoc—assumptions about equilibrium selection. Section 3 outlines the model. Section 4 analyzes the laissez-faire economy and derives a variety of conditions that ensure uniqueness of the symmetric switching equilibrium. In Section 5 we solve the social planner's problem; we discuss the implementation through time-varying taxation

[1] Greenspan's reasoning in our model should thus be rephrased as: "As I cannot know whether the public news is true or not, I don't want to intervene in the economy". We come to a different conclusion *despite* our assumption that policy makers receive no private information.

in Section 6. Finally, in Section 7 we discuss possible extensions and variants of our model, and some shortcomings of our approach.

2 Literature Review

Social learning has been intensively studied when players are assumed to move in an exogenously specified order.[2] Hendricks and Kovenock (1989) were the first to analyze a game with information externalities in which players choose whether and when to drill. They were also the first to highlight the possibility of an informational cascade: If Player one did not drill at time one, this signals unfavorable private information. In turn, this induces Player two not to drill at time two. In equilibrium both players may end up not drilling even though—had they pooled their private information—at least one player should have drilled at time one.

Although numerous papers analyze different waiting games,[3] to the best of our knowledge only Gossner and Melissas (2006), Levin and Peck (2008) and Doyle (2010) study optimal taxation in such a game. Furthermore, no paper in this literature analyzes the relationship between prior public information and optimal taxation. To fix ideas, motivate our modeling choices, highlight the novel contribution as well as compare our paper to existing ones, consider the following stylized social-learning setup: N players must decide whether or not to invest in a project, the cost of which is denoted by c. The returns of the project depend on the realized state of the world. If the state of the world is "high", the investment project yields a revenue equal to one. If the state of the world is "low", the project is assumed to yield zero revenues. Players receive a binary signal concerning the realized state of the world. Call a player who received a "low" signal a low-type player, while a high-type player received a "high" signal. After receiving their signals, players compute their posteriors. Let the posterior of a low-type player be denoted by μ_l while μ_h denotes the one of a high type. Suppose $c < \mu_l < \mu_h$. This

[2] For an excellent overview, see Chamley (2004b).

[3] Waiting games have, among others, also been analyzed by Chamley and Gale (1994), Gul and Lundholm (1995), Zhang (1995), Choi (Section 4, 1997), Caplin and Leahy (1998), Frisell (2003), and Gunay (2008).

parameter configuration either occurs because the investment cost is low, or because of a "favorable" prior. At time one, players simultaneously decide whether to invest or wait. If a player waits, she observes how many other players invested at time one and takes a final investment decision at time two. If a player invests at time two, however, her payoff gets discounted.

This set-up is plagued by multiple-equilibria. In one equilibrium, which is analyzed in the seminal paper of Chamley and Gale (1994), high types randomize between investing and waiting while low types wait.[4] As high types do not internalize their information externalities, Gossner and Melissas (2006) have shown that in this equilibrium investments should be *subsidized*.[5] Recall that $c < \mu_l$. As a low-type player also faces a positive gain from investing, it is a best reply for her to invest (at time one) if she expects all other $N - 1$ players to invest as well. Hence, there also exists another equilibrium in which all players invest at time one. In this equilibrium, "too many" types are investing and Gossner and Melissas highlight that a social planner can then raise welfare by *taxing* investments. The intuition should be clear: Through an appropriate investment tax, a social planner can reduce the profitability of investing such that only high types face a positive gain from investing. In that case, low types wait and benefit from the information externality. A model with a binary state and signal space is thus unable to generate unambigious economic policy recommendations when either public information is very conducive to investing—i.e. when the prior mean is "favorable"—or when the investment cost is "low". Below we derive such unambiguous policy recommendations by replacing the unrealistic assumption of binary returns to the investment project with the in our view more plausible assumption that the returns to investing are normally distributed. We show that policymakers should tax investments when the public sentiment is that

[4] Strictly speaking, Chamley and Gale do not prove that it is optimal for low types to wait. Instead, they assume that low types do not possess an "investment option" and therefore cannot invest. But giving low types the option to invest does not destroy their equilibrium.

[5] Doyle (2010) introduces idiosyncratic investment costs in such a setup and—following Chamley and Gale (1994)—assumes that low types cannot invest. High types invest if their investment costs lie below some critical level. In his model, the government cannot commit to a future tax/subsidy scheme. Players might thus postpone their investment plans in the hope to enjoy higher subsidies in the future. Doyle also finds that investments should be subsidized.

the investment opportunities are highly beneficial.

Chamley (2004a) analyzes a two-player continuous (and unbounded) signals version of the binary return-to-investment model and establishes the existence of multiple symmetric switching equilibria. Furthermore, Chamley (2004b) establishes that equilibrium is unique if the discount factor is sufficiently high. He does not, however, investigate the optimal tax policy, nor does he provide other sufficient conditions that guarantee uniqueness of equilibrium.

Levin and Peck (2008) introduce an idiosyncratic investment cost in the binary-return-to-investment set-up. In a two-player economy a small investment subsidy can lower welfare: Intuitively, a small subsidy can encourage some types with bad private information but low investment costs to invest, which can reduce the informational value of observing overall investment activity. They do not, however, provide conditions that guarantee the symmetric switching equilibrium is unique nor do they analyze the relationship between public information and tax policy, which is the focus of the current paper. With a continuum of investors, either a "laissez-faire" policy is optimal or investments should be subsidized. Intuitively, with a continuum learning is perfect as long as a positive fraction of players invest. In that case, the optimal subsidy is zero. If absent the subsidy no player invests, the government can benefit from subsidizing investments to induce a positive fraction of players to invest and thereby reveal the state of the world to all agents in the economy.

Recently, a global games approach (see Morris and Shin (2003) for a survey) has been developed to overcome multiplicity of equilibrium outcomes in various coordination settings—often with the aim of deriving policy recommendations. This approach typically consists in enriching the type and state space and assuming that some "extreme" types possess a "dominant strategy". *Some* authors then derive sufficient conditions that guarantee a unique equilibrium outcome, while others reduce the set of equilibrium outcomes. To keep the analysis tractable, many authors[6] enrich the type and state space by working with normally distributed random variables. As Vives (2005, JEL,

[6] See, among others, Angeletos, Hellwig and Pavan (2007) Angeletos and Werning (2006), Dasgupta (2007) and Morris and Shin (1999, 2000, 2002, 2003, 2004, 2005).

pages 471-472) points out, all papers reduce the set of equilibrium outcomes because the enrichment of the type and state space reduces the strength of the strategic complementarity. In this paper, we also work with normally distributed random variables when enriching the state and type space with the aim of predicting a unique symmetric switching equilibrium. This enables us to derive clear policy recommendations for the case in which public information is sufficiently favorable.

So far, only Dasgupta (2007) uses a global game approach in a social learning environment with the aim of predicting a unique symmetric switching equilibrium. He considers a two-period irreversible investment model with a continuum of players, exogenous observation noise, and positive network externalities. Dasgupta's paper focuses on how the ability to wait influences the extent of coordination failures in environments with positive network externalities and private information. He does not investigate the relationship between public information and optimal tax policy.

3 The Model

Two risk-neutral players have the possibility to invest in a risky project. Players can invest in two periods. If Player i invests at time one, she gets a monetary payoff of $\theta - \tau$. Henceforth, we refer to $\theta \in \mathbb{R}$ as the state of the world and $\tau \in \mathbb{R}$ as a temporary investment tax ($\tau > 0$) or subsidy ($\tau < 0$). If Player i invests at time two, she gets $\delta\theta$, where $\delta \in (0, 1)$ denotes the common discount factor. Investments are irreversible. The state of the world θ is randomly drawn from a normal distribution with mean $\bar{\theta}$ and variance σ_θ^2. The prior mean $\bar{\theta}$ could incorporate past public information. For example, $\bar{\theta}$ could be high because of an investment "boom" in the previous period(s). Relatedly, $\bar{\theta}$ could be high because many "stories" are circulating that depict the investment opportunity as a "golden" one. Player i receives a normally distributed private signal s_i concerning θ's realization. More precisely, we assume that $s_i = \theta + \epsilon_i$, where ϵ_i is independently drawn from a normal distribution with zero mean and variance σ_ϵ^2.

The timing is as follows: At time zero, the government sets the period-one investment tax τ. Thereafter, our waiting game starts with nature drawing the state of the world and all signals. After observing the investment tax τ and their private signals, players at period 1 simultaneously decide whether to invest or wait. At the beginning of period 2, players observe past investment choices. Any player who has not invested in period 1 then decides whether or not to invest in period 2. Finally, players receive their payoffs and the game ends.

We refer to the expected state of the world conditional on a player's signal as the player's time-one posterior mean, i.e. $\mu_i \equiv E(\theta|s_i)$. Throughout we exclusively focus on equilibria in symmetric switching strategies. Player i is said to follow a switching strategy if she invests at time one whenever her time-one posterior mean exceeds a critical threshold value μ^c and refrains from investing otherwise. A pair of strategies is a symmetric switching equilibrium if, given that Player j follows a switching strategy with critical threshold μ^*, one has: (E1) it is strictly optimal for Player i to invest in period one if and only if $\mu_i > \mu^*$; and (E2) if Player i did not invest at time one, she does so at time two if and only if her expectation of θ (given μ_i and given Player j's time-one decision) is positive.[7] Below equilibrium more generally refers to Bayesian equilibrium.[8] Suppose both players invest at time one if their posterior means exceed μ^c. Type μ_i is then said to be *active* at time two, if $\mu_i < \mu^c$.

4 Existence and Uniqueness of Switching Equilibria in a Laissez-Faire Economy

In this section we characterize equilibrium cutoffs when the government does not intervene in the economy, i.e. when $\tau = 0$. Let μ^{LF} denote the first-period *equilibrium* cutoff in a laissez-faire economy. We first characterize properties of the best response

[7] In Heidhues and Melissas (2010) we also prove that if the prior mean $\bar{\theta}$ is high enough, no asymmetric equilibrium in switching strategies exists.

[8] In our model players with sufficiently high (low) signals strictly prefer to invest (wait) at time one, independent of the other player's strategy. Hence, there are no off-the-equilibrium-path observations and players can always apply Bayes's rule so that any Bayesian equilibrium is consistent and sequentially rational.

to a switching strategy. To do so, it is useful to consider the expected payoff difference between investing early and delaying the investment decision. Let $\Delta(\mu_i, \mu_j^c)$ denote the difference between the gain of investing in period 1 and the gain of waiting as a function of Player i's posterior mean μ_i under the assumption that Player j follows a switching strategy characterized by μ_j^c. Thus,

$$\Delta(\mu_i, \mu_j^c) = \mu_i - \delta \Pr(\mu_j > \mu_j^c | \mu_i) \max\{0, E(\theta | \mu_i, \mu_j > \mu_j^c)\} - \delta \Pr(\mu_j < \mu_j^c | \mu_i) \max\{0, E(\theta | \mu_i, \mu_j < \mu_j^c)\}. \quad (1)$$

If $\Delta(\cdot) > 0$ Player i prefers to invest, while if $\Delta(\cdot) < 0$ she prefers to wait. We first observe that a player who is more optimistic regarding the state of the world has a bigger incentive to invest early. Formally,

Lemma 1. *A player's incentive to invest early increases in her time-one posterior mean, i.e.*

$$\frac{\partial \Delta(\mu_i, \mu_j^c)}{\partial \mu_i} > 0, \forall \mu_j^c.$$

Lemma 1 states a common property of waiting games studied in the literature.[9] Intuitively, the higher i's time-one posterior mean, the higher the probability that it will be optimal for her to invest at time two. Due to discounting, this makes waiting less attractive.

Lemma 1 implies that there exists a unique time-one posterior mean at which Player i is indifferent between investing and waiting given that Player j follows a switching strategy characterized by μ_j^c. Formally, i's cutoff $\mu_i^I(\mu_j^c)$ is implicitly defined through the equation $\Delta(\mu_i^I, \mu_j^c) = 0$.[10]

Suppose $\mu_i > 0$ and that i expects j to always wait so that $\mu_j^c = \infty$. Then, of course, j's waiting decision bears no informational content. Thus the difference between the gain of investing early and the gain of waiting and investing late is $\Delta(\mu_i, \infty) = (1 - \delta)\mu_i > 0$. On the other hand, if $\mu_i < 0$ and Player i expects Player j to always wait, Player i

[9] See for example Hendricks and Kovenock (1989) and Chamley (2004b, Lemma 6.1, p. 124).

[10] The superscript "I" stands for "indifferent".

prefers not to invest. Hence, in this case i invests in the first period whenever her time-one posterior mean is greater than zero and refrains otherwise. Using a similar reasoning, if Player i expects j to always invest, j's investment decision has no informational content and thus $\mu_i^I(-\infty) = \mu_i^I(\infty) = 0$. Furthermore, mere inspection of Equation 1 reveals that i's best response cutoff μ_i^I is continuous in μ_j^c. Lemma 1 thus implies that the cutoff μ^{LF} characterizes a symmetric switching equilibrium if and only if $\mu_i^I(\mu^{LF}) = \mu^{LF}$, or equivalently, $\Delta(\mu^{LF}, \mu^{LF}) = 0$.[11] Graphically, μ^{LF} is the point at which $\mu_i^I(\mu_j^c)$ crosses the 45-degree line. Since $\mu_i^I(-\infty) = \mu_i^I(\infty) = 0$, and since μ_i^I is continuous in μ_j^c, a symmetric switching equilibrium exists.

We now investigate conditions that guarantee uniqueness. First, observe that a player who is indifferent between investing and waiting must face a positive gain of investing. This implies that $\mu^{LF} > 0$. Because $\mu^{LF} < E(\theta|\mu_i = \mu^{LF}, \mu_j > \mu^{LF})$ a player with time-one posterior mean μ^{LF} invests at time two after observing her fellow player investing. We next argue that if $\mu_i = \mu^{LF}$, Player i does not invest in period two after observing that Player j waited, i.e. $E(\theta|\mu_i = \mu^{LF}, \mu_j < \mu^{LF}) < 0$. Given that j follows a switching strategy, observing him investing rather than waiting must make i more optimistic.[12] Hence, if i wants to invest after having observed that j waited, she must also want to invest after having observed that j invested. In such a case she invests at time two independent of j's time-one action. Her expected gain of waiting therefore is $\delta\mu^{LF}$. She is then better off, however, investing at time one and receiving an expected payoff of μ^{LF}.

Given this observation, we say that Player i receives "good news" when she observes j investing. Using that a cutoff type invests in period two only when receiving good news, $\Delta(\mu^{LF}, \mu^{LF})$ simplifies to

$$\Delta(\mu^{LF}, \mu^{LF}) = \mu^{LF} - \delta \Pr(\mu_j > \mu^{LF}|\mu_i = \mu^{LF}) E(\theta|\mu_i = \mu^{LF}, \mu_j > \mu^{LF}) = 0. \qquad (2)$$

Our analysis below makes use of some intuitive and well-known properties of the normal

[11] It follows from Lemma 1 that E1 is satisfied when $\Delta(\mu^{LF}, \mu^{LF}) = 0$. E2 is satisfied as well as Equation 1 prescribes Player i to make an optimal time-two choice.

[12] Formally, $E(\theta|\mu_i = \mu^{LF}, \mu_j > \mu^{LF}) > E(\theta|\mu_i = \mu^{LF}, \mu_j < \mu^{LF})$.

distribution (see the Appendix for proofs). First, Player i's first-period posterior mean μ_i is computed as:

$$\mu_i = \alpha s_i + (1-\alpha)\bar{\theta} \qquad \text{where} \qquad \alpha = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\epsilon^2}.$$

In words, μ_i is a weighted average between her private signal s_i and the prior mean $\bar{\theta}$. The more precise the prior information—i.e. the lower σ_θ^2—the more weight Player i puts on the prior mean and the less weight she puts on her signal. Conversely, the more precise her private information—i.e. the lower σ_ϵ^2—the more she trusts her signal as opposed to the prior mean. In particular, this implies that if the variance of the prior is infinite, or if the variance of her signal is zero, her posterior mean is equal to her signal.

Second, Player i's expectation of Player j's posterior mean μ_j is computed as:

$$E(\mu_j|\mu_i) = \alpha\mu_i + (1-\alpha)\bar{\theta}.$$

Intuitively, Player i believes that j's signal is distributed around her best guess of the true state of the world—i.e. her posterior mean. Player i, however, also realizes that Player j's posterior mean is a weighted average between j's signal and the prior mean, and therefore is likely to lie between i's posterior and the prior mean. Based on this, a key fact we use below is that if Player i's posterior mean increases by one unit, her expectation about j's posterior mean increases by *less* than one unit. Hence, for example, the further her posterior mean lies above the prior mean, the more likely i thinks that j is more pessimistic than herself. Closely related, if the signal is (nearly) perfect—i.e. the variance of the signal is (close to) zero—both players possess (almost) the same posterior. In that case Player i believes that she always (almost) lies in the "center of the world"—i.e. independent of her posterior there is a 50% chance of j being more optimistic than herself. A similar argument also applies with a completely uninformative prior—i.e. when the variance of the prior is infinite. In this case j puts zero weight on the prior mean when computing his posterior. As i believes j's signal to be distributed around her posterior mean, she also always believes to lie in the center of the world.

Third, conditional on having the cutoff posterior mean μ^{LF}, the probability that j's posterior is greater than the cutoff is

$$\Pr(\mu_j > \mu^{LF} | \mu_i = \mu^{LF}) = 1 - F\left(\kappa_1 \left(\mu^{LF} - \bar{\theta}\right)\right), \tag{3}$$

where F denotes the cumulative distribution function of the standard normal and where κ_1 is a positive constant depending on the prior and signal variances. It follows from our second observation as well as the formula above that an increase in $\mu^{LF} - \bar{\theta}$ reduces the probability of j being more optimistic than the cutoff type i.

Fourth, we are interested in the cutoff type's expectation about the state of the world when waiting and receiving good news. In a symmetric switching equilibrium, Player i's expectation will be based on her own signal, the prior mean, and the fact that j invested and thus had a first-period posterior mean above the common cutoff μ^{LF}. Here our distributional assumptions allow us to use known properties of the truncated normal distribution. Formally, in the Appendix we establish that

$$E(\theta | \mu_i = \mu^{LF}, \mu_j > \mu^{LF}) = \mu^{LF} + \kappa_2 h\left(\kappa_1 \left(\mu^{LF} - \bar{\theta}\right)\right), \tag{4}$$

where κ_2 is a positive constant which (just as κ_1) depends on σ_θ^2 and σ_ϵ^2, and where h represents the hazard rate of the standard normal distribution. Recall that the hazard rate h is defined here as: $h(x) \equiv \frac{f(x)}{1-F(x)}$.[13] Recall also that the hazard rate of a normal distribution is increasing in x. Intuitively, Player i's second-period expectation is the first-period expectation about the state of the world plus an upward shift that depends on the cutoff, the prior mean, as well as—through the constants—the variance of signals and the prior. We have seen above that the cutoff type's probability of getting good news decreases in the cutoff μ^{LF}. The above formula reveals that the impact of good news is also higher for higher cutoffs. Formally, this follows from the fact that the hazard rate of the standard normal distribution is increasing and thus, the upward shift is greater. The statistical intuition is as follows: Player i's belief of Player j's first-period posterior mean is normally distributed with—as we observed above—a mean that lies between i's posterior mean and the prior mean. As the cutoff increases, the expectation of Player

[13] Throughout the paper, f denotes the p.d.f. of a standard normal distribution.

j's posterior mean increases by less than the cutoff. Thus, if j invests he reveals that he lies in a higher quantile of this distribution. Since the expectation of a left-truncated normally distributed variable is increasing in the truncation point, the higher the cutoff the better the news for the cutoff type when observing j investing. Consider now the case in which the variance of the prior goes to infinity. As explained above, Player i then believes that she is in the "center of the world", i.e. there is, independent of her posterior, a 50 percent chance that j possesses a higher posterior than herself. This implies that the upward shift does not depend on the cutoff μ^{LF}. Mathematically, in the Appendix we show that κ_1 tends to zero as the variance of the prior goes to infinity, while κ_2 converges to a positive constant. Thus in this special case the upward shift is independent of where the cutoff lies.

Using Equations 3 and 4, we rewrite the equilibrium condition 2 as

$$\underbrace{\mu^{LF}}_{\text{Gain of investing}} = \underbrace{\delta \underbrace{\left[1 - F\left(\kappa_1(\mu^{LF} - \bar{\theta})\right)\right]}_{\text{Prob of good news}} \left[\mu^{LF} + \underbrace{\kappa_2 h(\kappa_1(\mu^{LF} - \bar{\theta}))}_{\text{Upward shift in beliefs}}\right]}_{\text{Discounted gain of waiting}}.$$

As μ^{LF} increases, there are two countervailing forces affecting the gain of waiting. On the one hand, the probability of getting good news decreases. On the other hand, as μ^{LF} increases receiving good news leads to a greater upward shift in beliefs. Indeed the expected upwards shift $[1 - F(\cdot)]\kappa_2 h(\cdot) = \kappa_2 f(\cdot)$ and therefore is non-monotone and unimodal. Rearranging by moving the linear terms in μ^{LF} to the left-hand side and rewriting, yields

$$\mu^{LF} = \kappa_2 \mathcal{X}\left(\kappa_1\left(\mu^{LF} - \bar{\theta}\right)\right), \text{ where } \mathcal{X}(\cdot) \equiv \frac{\delta f(\cdot)}{1 - \delta(1 - F(\cdot))}. \tag{5}$$

The left-hand side is linear in μ. The right-hand side is positive and goes to zero as μ goes to plus or minus infinity. Furthermore, Lemma 3 in the Appendix formally establishes many properties of our $\mathcal{X}$-function that are intuitive given that its numerator is the p.d.f. of a normally distributed random variable. In particular, we prove that $\mathcal{X}$ is unimodal, convex and increasing up to a critical value μ' and thereafter concave and increasing up

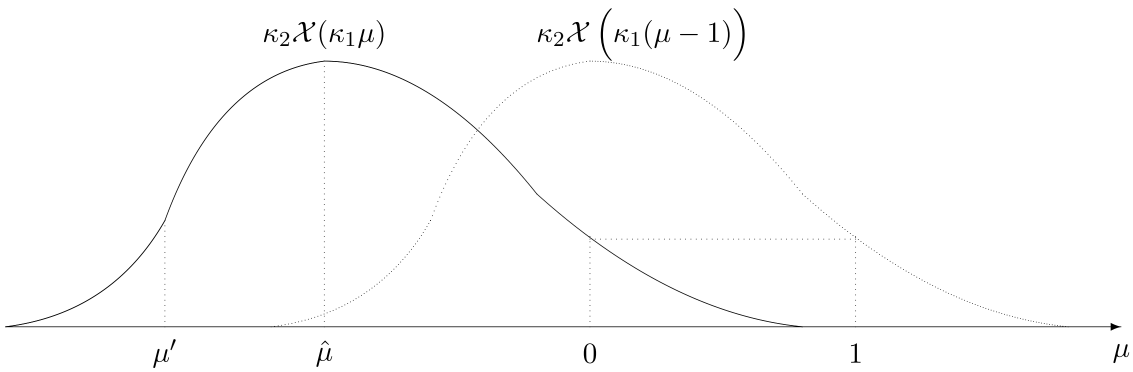

Figure 1: Shape of $\kappa_2\mathcal{X}(\kappa_1(\mu-\bar{\theta}))$ for $\bar{\theta}=0$ and $\bar{\theta}=1$.

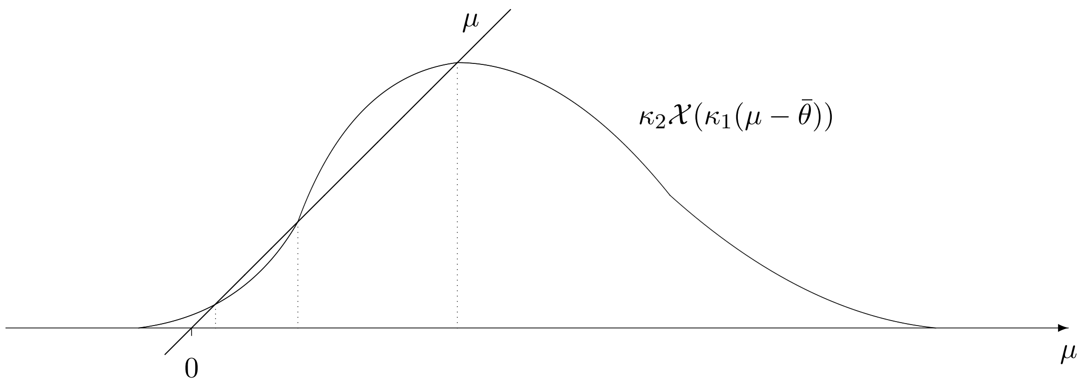

Figure 2: Three different equilibria.

to its mode $\hat{\mu}$. It is also easy to see that a unit increase in $\bar{\theta}$ leads to a translation to the right of $\mathcal{X}$ by one unit. This property is easiest to check when $\bar{\theta}$ increases from zero to one. In that case $\mathcal{X}(\kappa_1(0-0)) = \mathcal{X}(\kappa_1(1-1))$ as illustrated in Figure 1.[14]

As Figure 2 illustrates, whenever the slope of $\kappa_2\mathcal{X}$ is greater than one, multiple equilibria can arise. Intuitively, a low cutoff can be self-fulfilling since if μ^{LF} is low an agent's expected upward shift is also low; this makes waiting unattractive and thus induces players with low posterior means to invest early. If agents, however, expect a higher cutoff the expected upward shift can be higher, making waiting in turn more attractive.

Recall that if the variance of the prior is (infinitely) large, i believes j's posterior mean to be equally likely to lie above or below hers—independent of her posterior mean. The cutoff type's expected upward shift in this case is thus independent of her posterior

[14] At the risk of stating the obvious, $\mathcal{X}(\kappa_1(0-0))$ denotes the value of $\mathcal{X}(\kappa_1(\mu-\bar{\theta}))$ when $\mu=\bar{\theta}=0$.

mean. Hence, as the variance of the prior becomes large, the expected upward shift tends towards a constant and therefore the slope of $\kappa_2\mathcal{X}$ tends to zero. But whenever the slope of $\kappa_2\mathcal{X}$ is less than one everywhere, Figure 2 implies that symmetric switching equilibrium must be unique. Thus, for a high enough variance of the prior, equilibrium is unique. Similarly, as the agent's signal becomes infinitely precise (i.e. as $\sigma_\epsilon^2 \to 0$) she believes that she is in the center of the world and the expected upward shift tends to a constant. Thus, the symmetric switching equilibrium is also unique in this case. Furthermore, if the future becomes heavily discounted the gain of waiting and the slope of $\kappa_2\mathcal{X}$ tend to zero, and thus the unique equilibrium cutoff approaches zero in this case.

Of course, even if the maximal slope of $\kappa_2\mathcal{X}$ is greater than one, the symmetric switching equilibrium may be unique. For example, if the gain of investing μ crosses the function $\kappa_2\mathcal{X}$ in its right tail, i.e. when its slope is negative, the symmetric switching equilibrium is unique. Similarly, if it crosses $\kappa_2\mathcal{X}$ where its slope is positive but sufficiently low, the symmetric switching equilibrium will be unique. We have argued above that a unit increase in $\bar{\theta}$ leads to a translation by one unit to the right of $\kappa_2\mathcal{X}$. Hence, one can reduce $\bar{\theta}$ until the equilibrium condition 5 is satisfied in the decreasing part of $\kappa_2\mathcal{X}$. Similarly, we can increase $\bar{\theta}$ until μ cuts $\kappa_2\mathcal{X}(\cdot)$ "far enough" in its left tail. Thus for sufficiently high or sufficiently low $\bar{\theta}$, the symmetric switching equilibrium is unique. If the prior mean $\bar{\theta}$ even becomes arbitrarily large, μ cuts $\kappa_2\mathcal{X}(\cdot)$ when μ is arbitrarily close to zero. (This is illustrated in Figure 3.) Similarly, when the prior mean becomes arbitrarily negative, μ^{LF} also tends to zero.

Finally, we argue that the symmetric switching equilibrium is unique if players are very patient. To see the intuition, consider the limit case in which $\delta = 1$. In that case $\mathcal{X}(\kappa_1(\mu - \bar{\theta}))$ simplifies to the reverse hazard rate of the standard normal distribution.[15] Hence in the limit case of perfectly patient agents, μ and $\mathcal{X}(\kappa_1(\mu - \bar{\theta}))$ cross each other once, as μ is increasing and $\mathcal{X}(\kappa_1(\mu - \bar{\theta}))$ decreasing in μ. In the Appendix, we extend this argument by showing that $\mathcal{X}(\kappa_1(\mu - \bar{\theta}))$ is decreasing in the relevant range if players

[15] Recall that the reverse hazard rate of a standard normal distribution, r, is defined as: $r \equiv \frac{f(x)}{F(x)}$. As is well known, r is decreasing in x.

are sufficiently patient.

The following proposition summarizes the above discussion:[16]

Proposition 1. *If*

$$\frac{\partial \kappa_2 \mathcal{X}\left(\kappa_1(\mu - \bar{\theta})\right)}{\partial \mu} \leq 1, \qquad \forall \mu \tag{6}$$

there exists a unique symmetric switching equilibrium. Furthermore, Inequality 6 is satisfied if either:

1. $\sigma_\theta^2 > (\sigma_\theta^2)^c$ *for a given* $(\sigma_\theta^2)^c < \infty$, *or*

2. $\sigma_\epsilon^2 < (\sigma_\epsilon^2)^c$ *for a given* $(\sigma_\epsilon^2)^c > 0$, *or*

3. $\delta \leq \underline{\delta}$ *for a given* $\underline{\delta} > 0$.

If Inequality 6 is not satisfied, there exist values of $\bar{\theta}$ *that support multiple symmetric switching equilibria. The symmetric switching equilibrium, however, is still unique if either*

4. $\bar{\theta} \leq 0$, *or*

5. $\bar{\theta} \geq \bar{\theta}_u$ *for a given* $\bar{\theta}_u < \infty$.

Furthermore, for any given $\bar{\theta}$, *equilibrium is also unique if*

6. $\delta \geq \bar{\delta}$ *for a given* $\bar{\delta} < 1$.

Finally, $\lim_{\bar{\theta} \to -\infty} \mu^{LF} = \lim_{\bar{\theta} \to \infty} \mu^{LF} = 0$.

Dasgupta (2007) also analyzes a dynamic game with social learning (see our literature review for more details) and establishes uniqueness of the symmetric switching equilibrium if either condition one or two holds. In our two-player model without network externalities, we identify additional conditions that yield uniqueness. Chamley (2004b) analyzes a similar set-up as ours and shows that the symmetric equilibrium is unique if the discount factor is sufficiently high. As we work with normally distributed random variables, we were able to identify additional sufficient conditions.

[16] In the proposition, $(\sigma_\theta^2)^c$, $(\sigma_\epsilon^2)^c$, $\underline{\delta}$, $\bar{\theta}_u$, and $\bar{\delta}$ represent cutoff values all of which are functions of the exogenous parameters of our model.

In many applications, players observe each other frequently and can relatively quickly react upon observing a player's investment decision. Hence, in such situations the discount factor is high and our model yields a unique equilibrium. Similarly, "boom" times are typically characterized by "stories" that depict some investment opportunities as "golden" ones. Hence, in such a situation our model also yields a unique switching equilibrium even if players observe each others actions only infrequently.

5 The Social Planner's Problem

In this section, we consider a social planner who chooses three cutoff levels: a time-one cutoff μ^c above which a player invests at time one if (and only if) her time-one posterior mean exceeds it; a time-two cutoff $\underline{\mu}^1$ for the case in which her fellow player invested at time-one; and a time-two cutoff $\underline{\mu}^0$ for the case in which her fellow player did not invest at time one. At time two, a player invests if (and only if) her time-one posterior mean lies above the relevant time-two cutoff. The social planner aims to maximize expected welfare W, which is defined as

$$
\begin{aligned}
W \quad \equiv \quad & \int \Pr(\mu_i > \mu^c, \mu_j > \mu^c | \theta) 2\theta f\left(\frac{\theta - \bar{\theta}}{\sigma_\theta}\right) d\theta \\
+ \quad & 2\int \Pr\left(\mu_i > \mu^c, \mu_j \in \left[\min\{\underline{\mu}^1, \mu^c\}, \mu^c\right] \middle| \theta\right) (1+\delta)\theta f\left(\frac{\theta - \bar{\theta}}{\sigma_\theta}\right) d\theta \\
+ \quad & 2\int \Pr(\mu_i > \mu^c, \mu_j < \min\{\underline{\mu}^1, \mu^c\} | \theta) \theta f\left(\frac{\theta - \bar{\theta}}{\sigma_\theta}\right) d\theta \\
+ \quad & \delta \int \Pr\left(\mu_i \in \left[\min\{\underline{\mu}^0, \mu^c\}, \mu^c\right], \mu_j \in \left[\min\{\underline{\mu}^0, \mu^c\}, \mu^c\right] \middle| \theta\right) 2\theta f\left(\frac{\theta - \bar{\theta}}{\sigma_\theta}\right) d\theta \\
+ \quad & 2\delta \int \Pr\left(\mu_i \in \left[\min\{\underline{\mu}^0, \mu^c\}, \mu^c\right], \mu_j < \min\{\underline{\mu}^0, \mu^c\} \;\middle|\; \theta\right) \theta f\left(\frac{\theta - \bar{\theta}}{\sigma_\theta}\right) d\theta.
\end{aligned} \tag{7}
$$

The first integral captures the case in which both players invest at time one in which case welfare is equal to 2θ. The second integral captures the case in which one player invests at time one and thereby induces the other player to invest at time two. The player who invests at time one gets θ, the one who invests at time two $\delta\theta$. Observe that the second integral equals zero if $\mu^c \leq \underline{\mu}^1$.[17] The third integral captures the case in which only one player invests at time one. The other player's (time-one) posterior is too low and she therefore never invests. In the fourth integral both players invest at time two. Welfare then equals $2\delta\theta$. Observe also that the fourth integral equals zero if $\mu^c \leq \underline{\mu}^0$. Finally, in the last integral only one player invests at time two. One can think of $\frac{1}{2}W \equiv U$ as the expected utility of a representative player in our model.

It is useful to rewrite U in terms of the ex ante distribution of a player's posterior. From the planner's point of view, μ_i is normally distributed with mean $\bar{\theta}$ and with some variance denoted by σ_μ^2.[18] Using this, in the Appendix we show that the expected utility of the representative player can be rewritten as:

$$\begin{aligned} U &= \int_{\mu^c}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i \\ &+ \delta \int_{\min\{\underline{\mu}^0, \mu^c\}}^{\mu^c} \Pr(\mu_j < \mu^c | \mu_i) E(\theta | \mu_i, \mu_j < \mu^c) f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i. \\ &+ \delta \int_{\min\{\underline{\mu}^1, \mu^c\}}^{\mu^c} \Pr(\mu_j > \mu^c | \mu_i) E(\theta | \mu_i, \mu_j > \mu^c) f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i \end{aligned} \tag{8}$$

Equation 8 is intuitive: The first integral represents the weighted expected utility of all types that invest at time one. Our second integral captures Player i's payoff in case both players waited at time one; in this case a player invests if her time-one posterior mean lies in the interval $[\underline{\mu}^0, \mu^c]$ and she gets an expected payoff of $\delta E(\theta|\mu_i, \mu_j < \mu^c)$ when investing. The third integral captures the case in which Player i does not invest in period 1 but does so in period 2 when receiving good news ($\mu_i \in [\underline{\mu}^1, \mu^c]$), which in turn happens with probability $\Pr(\mu_j > \mu^c|\mu_i)$.

We are now ready to analyze the optimal time-two cutoffs. Clearly, welfare cannot be raised by obliging a player to forego a profitable investment opportunity at time two

[17] At the risk of stating the obvious, this integral is multiplied by two as this event happens either when $\mu_1 > \mu^c$ and $\mu_2 \in [\underline{\mu}^1, \mu^c]$ or when $\mu_2 > \mu^c$ and $\mu_1 \in [\underline{\mu}^1, \mu^c]$.

[18] In Section 4, we argued that i's posterior mean μ_i is a weighted average between her signal and the prior mean. Formally, $\mu_i = \alpha s_i + (1-\alpha)\bar{\theta}$ (where $\alpha \in [0,1]$ depends on the prior and signal variances). By assumption $s_i = \theta + \epsilon_i$, where $\theta \sim N(\bar{\theta}, \sigma_\theta^2)$ and $\epsilon_i \sim N(0, \sigma_\epsilon^2)$. As ϵ_i is independent of θ, from the planner's point of view $s_i \sim N(\bar{\theta}, \sigma_\theta^2 + \sigma_\epsilon^2)$. Hence, μ_i is the sum of a normally distributed random variable (multiplied by α) with mean $\bar{\theta}$ and a constant (i.e. $(1-\alpha)\bar{\theta}$). This implies that $\mu_i \sim N(\bar{\theta}, \sigma_\mu^2)$ where $\sigma_\mu^2 = \alpha^2(\sigma_\theta^2 + \sigma_\epsilon^2)$.

or by forcing a rational player to invest in the second period when she believes this to be unprofitable. Hence, welfare-maximization implies that the critical investment type when getting good news ($\underline{\mu}^1$) is implicitly defined by setting the expected second-period investment return to zero (i.e. through $E(\theta|\mu_i = \underline{\mu}^1, \mu_j > \mu^c) = 0$). Similarly, the critical investment type when getting bad news ($\underline{\mu}^0$) is implicitly defined through $E(\theta|\mu_i = \underline{\mu}^0, \mu_j < \mu^c) = 0$.[19] With a slight abuse of notation, $\underline{\mu}^0$ and $\underline{\mu}^1$ will henceforth denote the *optimal* time-two cutoffs. Note that $\underline{\mu}^0$ and $\underline{\mu}^1$ depend on the time-one cutoff μ^c.

For which time-one cutoff μ^c is it optimal to have some active players invest in period two? Consider the cutoff type μ^c. Suppose that the expected state of the world is negative for this cutoff type even when receiving good news, i.e. that $E(\theta|\mu_i = \mu^c, \mu_j > \mu^c) \leq 0$. As we established above, it is then optimal for the cutoff type to refrain from investing in period two when getting good news—and because the cutoff type is the most optimistic type who waits, no other type wants to invest in period two. Furthermore, as the expected state of the world is even lower when getting bad news, no type will want to invest in period 2 whenever $E(\theta|\mu_i = \mu^c, \mu_j > \mu^c) \leq 0$. Lemma 2 in the Appendix proves that there exists a unique lower bound $\underline{\mu}$ such that $E(\theta|\mu_i = \underline{\mu}, \mu_j > \underline{\mu}) = 0$, which implies that there is no time-two investments whenever $\mu^c < \underline{\mu}$. Because players become more optimistic when getting good news, it is obvious that the lower bound is negative, i.e. $\underline{\mu} < 0$.

When $\mu^c > \underline{\mu}$, the social planner instructs the cutoff type who receives good news to invest as her expectation of the realized state of the world is then positive ($E(\theta|\mu^c, \mu_j > \mu^c) > 0$). If μ^c, however, is close to $\underline{\mu}$, the expected state of the world when getting bad news is still negative for the cutoff type. Hence, no active type will be instructed to invest when getting bad news. When μ^c is high enough, even when getting bad news the expected state of the world is positive ($E(\theta|\mu^c, \mu_j < \mu^c) > 0$). Active types close

[19] Lemma 2 in the Appendix formally establishes that for any first-period cutoff μ^c, there exists a unique second-period cutoffs $\underline{\mu}^0$ and $\underline{\mu}^1$, and that the expectations $E(\theta|\mu_i, \mu_j > \mu^c)$ and $E(\theta|\mu_i, \mu_j < \mu^c)$ are increasing in μ_i. Hence, Player i should invest at time two if (and only if) her time-one posterior mean lies above the relevant cutoff.

enough to μ^c will then optimally invest at time two when getting bad news. As these types are even more optimistic when getting good news, they will also invest in that case. Lemma 2 in the Appendix proves the existence of a a unique upper bound $\bar{\mu}$ such that $E(\theta|\mu_i = \bar{\mu}, \mu_j < \bar{\mu}) = 0$, which implies that active types will refrain from investing when getting bad news if and only if $\mu^c < \bar{\mu}$. Since bad news makes a player more pessimistic, observe that $\bar{\mu} > 0$.

To summarize, if the social planner implements a very low cutoff (i.e. if $\mu^c < \underline{\mu}$), no one invests at time two and thus the utility of the representative agent is

$$\forall \mu^c \leq \underline{\mu}, \qquad U = \int_{\mu^c}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i.$$

If $\mu_i \in [\underline{\mu}, \bar{\mu}]$, on the other hand, some active types invest at time two if they receive good news but everyone refrains from investing when getting bad news. Hence,

$$\begin{aligned} \forall \mu^c \in [\underline{\mu}, \bar{\mu}], \quad U \;&=\; \int_{\mu^c}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i \\ &+\; \delta \int_{\underline{\mu}^1}^{\mu^c} \Pr(\mu_j > \mu^c|\mu_i) E(\theta|\mu_i, \mu_j > \mu^c) f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i. \end{aligned} \tag{9}$$

For $\mu^c > \bar{\mu}$, some active types will invest when getting either bad or good news while others invest only when getting good news.

Let μ^{SP} denote the first-period cutoff that maximizes U. We now argue that $\mu^{SP} > \underline{\mu}$. Suppose otherwise, i.e. that $\mu^{SP} \leq \underline{\mu} < 0$. In this case the expected utility of the representative player equals

$$\int_{\mu^{SP}}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i.$$

Because $\mu^{SP} < 0$, the social planner can raise welfare by setting $\mu^{SP} = 0$—a contradiction. Thus $\mu^{SP} > \underline{\mu}$. This result allows us to prove that the social planner will choose a higher than the laissez-faire cutoff when the prior mean is sufficiently high.

Recall from our previous section that, in a Bayesian equilibrium without taxes, no active type invests at time two if no one invested at time one, i.e. $\mu^{LF} < \bar{\mu}$.[20] Fur-

[20] Using the same reasoning as in Section 4, it is straightforward to prove that the inequality ($\mu^{LF} < \bar{\mu}$) is strict: Suppose $\mu^{LF} = \bar{\mu}$. If the cutoff type μ^{LF} receives bad news, she is indifferent between investing and not investing. One can thus think of her as investing at time two—independent of the other player's decision. But then she is better off investing early and saving on the discounting cost.

thermore, $\mu^{LF} > 0$ as players with a non-positive posterior mean prefer to wait. Hence, $\mu^{LF} \in (\underline{\mu}, \bar{\mu})$. It follows from Equation 9 that $\forall \mu^c \in [\underline{\mu}, \mu^{LF}]$,

$$\begin{aligned}
\frac{dU}{d\mu^c} &= -\Big[\mu^c - \delta \Pr(\mu_j > \mu^c|\mu^c) E(\theta|\mu^c, \mu_j > \mu^c)\Big] f\left(\frac{\mu^c - \bar{\theta}}{\sigma_\mu}\right) \\
&- \delta \frac{d\underline{\mu}^1}{d\mu^c} \Pr(\mu_j > \mu^c|\underline{\mu}) \underbrace{E(\theta|\underline{\mu}^1, \mu_j > \mu^c)}_{=0} f\left(\frac{\underline{\mu}^1 - \bar{\theta}}{\sigma_\mu}\right) \\
&+ \delta \underbrace{\int_{\underline{\mu}^1}^{\mu^c} \frac{\partial}{\partial \mu^c} \Big(\Pr(\mu_j > \mu^c|\mu_i) E(\theta|\mu_i, \mu_j > \mu^c)\Big) f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i}_{\text{Weighted change in the inframarginal types' gain of waiting}}.
\end{aligned} \tag{10}$$

For any given first-period cutoff, recall that the socially optimal cutoff $\underline{\mu}^1$ is implicitly defined through $E(\theta|\mu_i = \underline{\mu}^1, \mu_j > \mu^c) = 0$. Hence, the second term of the right-hand side equals zero. The term between square brackets represents $\Delta(\mu^c, \mu^c)$, i.e. the difference between the cutoff type's gain of investing early and her gain of waiting. Suppose that $\mu^c = \underline{\mu} < 0$. By definition of $\underline{\mu}$ this means that the cutoff type μ^c is indifferent between investing and not investing in case she gets good news. Hence, one can think of her as someone who will not invest at time two—independent of the other player's time-one decision. In that case her gain of waiting is zero and $\Delta(\underline{\mu}, \underline{\mu}) = \underline{\mu} < 0$. Suppose now that $\mu^c = \mu^{LF}$. As the cutoff player is indifferent between investing and waiting, $\Delta(\mu^{LF}, \mu^{LF}) = 0$. If equilibrium is unique—as is the case if the prior mean is sufficiently high[21]—there exists no other cutoff $\mu^c \neq \mu^{LF}$ such that $\Delta(\mu^c, \mu^c) = 0$. By continuity, the term between square brackets (i.e. *excluding* the minus sign in front) is thus negative if equilibrium is unique and if $\mu^c \in [\underline{\mu}, \mu^{LF})$. Hence, if the prior mean is sufficiently high, and if the social planner implements a cutoff $\mu^c < \mu^{LF}$, then the first term on the right-hand side (i.e. *including* the minus sign in front) is positive. Using 10, we thus conclude that $\forall \mu^c \in [\underline{\mu}, \mu^{LF}]$, $dU/d\mu^c > 0$ if the prior mean $\bar{\theta}$ is sufficiently high and if

$$\int_{\underline{\mu}^1}^{\mu^c} \frac{\partial}{\partial \mu^c} \Big[\Pr(\mu_j > \mu^c|\mu_i) E(\theta|\mu_i, \mu_j > \mu^c)\Big] f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i > 0. \tag{11}$$

We now argue that, if the prior mean $\bar{\theta}$ is high enough, the above inequality is satisfied. To build some intuition, we first explain how Player i's gain of waiting is influenced by

[21] See Proposition 1.

the cutoff μ^c. Suppose $\mu_i \in [\underline{\mu}^1, \mu^c]$ in which case Player i invests at time two if the other player did so at time one. In the Appendix (see proof of Proposition 2), we prove that

$$\frac{\partial}{\partial \mu^c}\Big[\Pr(\mu_j > \mu^c|\mu_i)E(\theta|\mu_i, \mu_j > \mu^c)\Big] > 0 \qquad \Leftrightarrow \qquad (1-\alpha)\bar{\theta} - \mu_i > \mu^c. \tag{12}$$

Stated differently, Player i's gain of waiting is unimodal: It increases until $\mu^c = (1-\alpha)\bar{\theta} - \mu_i$ and decreases thereafter. To understand the unimodality of Player i's gain of waiting, observe that the above derivative is equal to

$$\frac{\partial \Pr(\mu_j > \mu^c|\mu_i)}{\partial \mu^c}E(\theta|\mu_i, \mu_j > \mu^c) + \Pr(\mu_j > \mu^c|\mu_i)\frac{\partial E(\theta|\mu_i, \mu_j > \mu^c)}{\partial \mu^c}.$$

The first term of this sum is negative while the second one is positive. Recall from our discussion of Equation 4 that a player's expectation of the state of the world when receiving good news is the sum of her time-one posterior mean and an upward shift. Furthermore, if the critical time-one cutoff μ^c is low, both μ^c and the upward shift are small, and hence the first term in the above sum is not very negative. In addition, if μ^c is low, it is very likely that the other player will invest early. Hence, any increase in Player i's upward shift is multiplied by a large number, so that the second term in above sum is large. Hence, if μ^c is low the above derivative is positive. In words, if μ^c is low, Player i wants the social planner to raise the cutoff as the decrease in the probability of getting good news is more than compensated by the increase in her upward shift. The contrary situation prevails when μ^c is high: Player i then prefers the cutoff to be lowered in order to increase her probability of getting good news. Call μ_i^{max} the cutoff (above which Player j invests) that maximizes Player i's gain of waiting. It follows from 12 that

$$\mu_i^{max} = (1-\alpha)\bar{\theta} - \mu_i. \tag{13}$$

Suppose Player j invests. Player i then knows that $\mu_j > \mu^c$ and computes $E(\mu_j|\mu_i, \mu_j > \mu^c)$. (Keeping μ_i fixed, the higher this expectation, the higher her expectation of the realized state of the world θ.) Recall from our previous section that $\mu_j|\mu_i$ is normally distributed with mean $\alpha\mu_i + (1-\alpha)\bar{\theta}$. Thus the higher $\bar{\theta}$, the lower the quantile in which Player j's posterior mean in the p.d.f. of $\mu_j|\mu_i$ lies, and the lower Player i's upward shift. If the prior mean $\bar{\theta}$ increases, Player i thus prefers Player j's cutoff to also increase to avoid any dilution of her upward shift. Observe that Player i's upward shift is also diluted by an increase in μ_i. Yet, Equation 13 reveals that μ_i^{max} is *decreasing* in her posterior mean μ_i. To understand this, recall that Player i only invests at time two if Player j did so at time one. Hence, the higher μ_i the more Player i "fears" that Player j will not invest at time one. Stated differently, if μ_i increases Player i prefers the cutoff μ^c to be reduced even if this comes at the cost of a lower upward shift.

Suppose now that

$$(1-\alpha)\bar{\theta} - \mu^{LF} > \mu^{LF}. \tag{14}$$

Economically, this inequality states that the cutoff-type (of the Bayesian equilibrium in a laissez-faire economy) thinks that the cutoff μ^{LF} is too low. Hence, if this condition holds the gain of waiting of the cutoff type $\mu_i = \mu^{LF}$ increases if the social planner increases the first-period cutoff.

In Section 4 we established that μ^{LF} tends to zero when the prior mean $\bar{\theta}$ goes to infinity. Hence, Inequality 14 is satisfied if the prior mean $\bar{\theta}$ is sufficiently high. Thus, the marginal type—and by the argument above all inframarginal types—think that the cutoff μ^{LF} is too low. Hence, if 14 is satisfied, then Inequality 11 is also satisfied and we conclude:

PROPOSITION 2. *If the prior mean $\bar{\theta}$ is high enough, the social planner's optimal period-one investment cutoff is strictly higher than in the laissez-faire economy $\mu^{SP} > \mu^{LF}$. Furthermore, any period-one investment cutoff $\mu^c < \mu^{LF}$ yields a lower welfare than the one which prevails in a laissez-faire economy; finally, there exists an $\epsilon > 0$ such that for all time-one cutoffs μ^c satisfying $\mu^c - \mu^{LF} \in (0, \epsilon)$, welfare is strictly greater than in the laissez-faire economy.*

Proposition 2 extends intuition about the insufficient use of private information derived in the original herding papers (see Banerjee (1992) and Bhikhchandani et al. (1992)) to an endogenous queue set-up. In an informational cascade, Player i gets say sufficiently good public information about the returns from investing, which arises when enough predecessors in a queue decide to invest, so that she follows the public information and invests even when possessing an unfavorable private signal. This investment decision is typically socially inefficient as it impedes subsequent movers to infer this player's signal from her action. A similar inefficiency also arises in our model: when stories about the high profitability of an investment opportunity abound (i.e. if $\bar{\theta}$ is sufficiently high), an inefficiently high mass of types end up investing early, making it harder for players who wait to confidently infer that the state of the world is indeed conducive to investing.

We now argue that, if the prior mean becomes arbitrarily negative, $\mu^{SP} \leq \mu^{LF}$, i.e. in the limit the laissez-faire cutoff is not "too low". As explained above, Player i's gain of waiting is unimodal and maximal when Player j invests if (and only if) $\mu_j > (1-\alpha)\bar{\theta}-\mu_i$. As explained in our previous section, μ^{LF} tends to zero as the prior mean $\bar{\theta}$ goes to minus infinity. Hence, for $\bar{\theta}$ sufficiently low,

$$\underbrace{(1-\alpha)\bar{\theta}-\mu^{LF}}_{\text{Type } \mu^{LF}\text{'s preferred cutoff}} < \underbrace{\mu^{LF}}_{\text{Equilibrium cutoff}},$$

which implies that types with posterior means close to (but less than) μ^{LF} think that μ^{LF} is *too high*. It follows from our discussion after Equation 13 that $\underline{\mu}^1$ is decreasing in the prior mean: The lower the prior mean, the higher the upward shift in Player i's posterior mean. Hence, when the prior mean becomes very negative some types with a very negative (time-one) posterior mean may end up investing at time two. Actually, it is simple to prove that $\underline{\mu}^1$ goes to minus infinity as the prior mean $\bar{\theta}$ also goes to minus infinity. In the Appendix (see proof of Proposition 3), however, we show that

$$\lim_{\bar{\theta}\to-\infty} \underbrace{\mu^{LF}}_{\text{Equilibrium cutoff}} = 0 = \lim_{\bar{\theta}\to-\infty} \underbrace{(1-\alpha)\bar{\theta}-\underline{\mu}^1}_{\text{Type } \underline{\mu}^1\text{'s preferred cutoff}}.$$

Hence, in the limit the inframarginal type with the lowest posterior (i.e. type $\underline{\mu}^1$) is perfectly happy with the cutoff μ^{LF}. It then follows from 13 that all the other inframarginal types (i.e. all types $\mu_i \in (\underline{\mu}^1, \mu^{LF}]$) think that the laissez-faire cutoff is too high. Their gain of waiting would be higher if the social planner were to implement a lower cutoff.

This result implies that if the social planner implements a cutoff $\mu^c > \mu^{LF}$, there exists some critical prior mean $\bar{\theta}^c$ such that if $\bar{\theta} < \bar{\theta}^c$ the social planner could raise welfare by reducing the cutoff μ^c. This result also implies that, as $\bar{\theta}$ goes to minus infinity, $\mu^{SP} \leq \mu^{LF}$. To summarize:

Proposition 3. *Any time-one cutoff $\mu^c > \mu^{LF}$ is suboptimally high if $\bar{\theta}$ lies below some threshold $\bar{\theta}^c(\mu^c)$.*

6 Implementation

We now discuss how a social planner that selects first- and second-period taxes can implement the optimal investment cutoffs. In our previous section we argued that the time-two cutoffs $\underline{\mu}^0$ and $\underline{\mu}^1$ for any given period-one cutoff should be chosen such that no profitable investment opportunity is wasted at time two, i.e. $E(\theta|\mu_i = \underline{\mu}^0, \mu_j < \mu^c) = 0$ and $E(\theta|\mu_i = \underline{\mu}^1, \mu_j > \mu^c) = 0$. Hence the optimal second-period tax is zero. For the remainder of the section, we focus on the first-period tax τ. A tax τ is said to implement a first-period cutoff μ^c if there exists *a* Bayesian equilibrium with the property that $\mu^* = \mu^c$.

We have established in Section 5 that the optimal cutoff μ^{SP} is not "very low" (i.e. $\mu^{SP} > \underline{\mu}$). In words, our last inequality means that the social planner will never set μ^c so low that investment activity stops prior to time two. Instead he will choose μ^c such that a meaningful information externality still remains, i.e. some active types who observe the other player investing at time one will be induced to invest at time two. We thus restrict attention to cutoffs $\mu^c > \underline{\mu}$. Note also that $\mu^{LF} < \bar{\mu}$ since in the laissez-faire equilibrium a period-one cutoff type invests in period two if and only if she gets good news, while for cutoffs above $\bar{\mu}$ a period-one cutoff type also invests in period two when getting bad news.

Suppose the social planner wants to implement a cutoff $\mu^c \in [\underline{\mu}, \bar{\mu}]$ so that the cutoff type μ^c invests at time two if and only if the other player did so at time one. Then the cutoff type is indifferent between investing and waiting if

$$\mu^c - \tau = \delta \Pr(\mu_j > \mu^c | \mu_i = \mu^c) E(\theta | \mu_i = \mu^c, \mu_j > \mu^c) \tag{15}$$

Recall from Equations 3 and 4 that

$$\Pr(\mu_j > \mu^c | \mu_i = \mu^c) = 1 - F\left(\kappa_1(\mu^c - \bar{\theta})\right), \tag{16}$$

and that

$$E(\theta | \mu_i = \mu^c, \mu_j > \mu^c) = \mu^c + \kappa_2 h\left(\kappa_1(\mu^c - \bar{\theta})\right). \tag{17}$$

Using 16 and 17, the indifference equation 15 can be rewritten as:

$$\mu^c - \frac{\tau}{1 - \delta\left(1 - F\left(\kappa_1(\mu^c - \bar{\theta})\right)\right)} = \kappa_2 \mathcal{X}(\kappa_1(\mu^c - \bar{\theta})). \tag{18}$$

The right-hand side of the indifference equation above was analyzed in Section 4. Observe that if $\tau = 0$, the indifference condition boils down to 5. Call LHS the left-hand side of the above equation after replacing μ^c with μ and observe that LHS is continuous in μ, and that it goes from minus to plus infinity as μ goes from $-\infty$ to $+\infty$. Observe also that LHS always lies above μ if τ is negative. If τ is positive, however, LHS always lies below μ. This implies that if equilibrium is unique in the laissez-faire economy, and if the social planner wants to raise the equilibrium cutoff (i.e. if $\mu^{SP} \in (\mu^{LF}, \bar{\mu})$), he must tax investments. (This case is illustrated in Figure 3.) Conversely, if equilibrium is unique in the laissez-faire case and the social planner wants to implement a cutoff $\mu^{SP} \in (\underline{\mu}, \mu^{LF})$, he must subsidize investments.

One can rewrite the equilibrium condition 18 as:

$$\tau = \left[1 - \delta\left(1 - F\left(\kappa_1(\mu^c - \bar{\theta})\right)\right)\right] \mu^c - \kappa_2 \delta f\left(\kappa_1(\mu^c - \bar{\theta})\right).$$

The social planner can thus implement any $\mu^c \in [\underline{\mu}, \bar{\mu}]$ by setting τ equal to the right hand side of the above equation.

Suppose now that the social planner wants to implement a cutoff $\mu^c > \bar{\mu} > \mu^{LF}$. It then follows from our previous section that the cutoff type μ^c will invest at time two—independent of the other player's time-one action. Hence, in this case the cutoff type

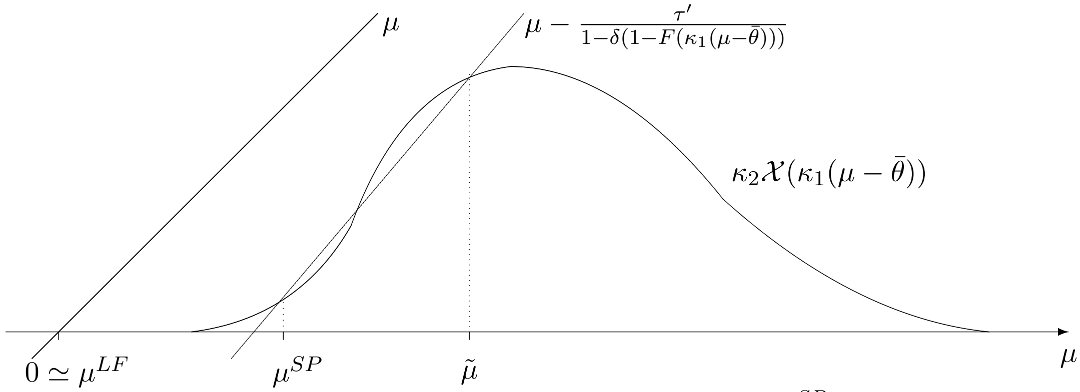

Figure 3: Non-unique implementation of μ^{SP}.

is indifferent between investing and waiting if $\mu^c - \tau = \delta\mu^c$. Any $\mu^c > \bar{\mu}$ can thus be implemented by choosing τ such that $\tau = (1-\delta)\mu^c > 0$, where the inequality follows from the fact that $\mu^c > \bar{\mu} > 0$.

An important caveat, however, is that a given tax τ need not uniquely implement μ^{SP}! To understand this, consider Figure 3. In the figure, the prior mean $\bar{\theta}$ is implicitly assumed to be "high". (This explains why in the figure the median of $\mathcal{X}$ is drawn so much to the right and—as explained in Section 4—why μ^{LF} is close to zero.) As summarized in Proposition 2, if the prior mean is high enough, the equilibrium cutoff μ^{LF} is too low. (This is illustrated in Figure 3 by the fact that $\mu^{LF} < \mu^{SP}$.) The figure also shows that if investments are appropriately taxed (i.e. if $\tau = \tau'$), there exists an equilibrium in which players coordinate on the efficient time-one cutoff μ^{SP}. In the figure, however, there also exists another equilibrium in which a player invests if and only if her time-one posterior mean exceeds $\tilde{\mu}$. In this case the investment tax τ' actually deters too many types from investing. If players focus on this latter equilibrium, the investment tax τ' may even *decrease* welfare as compared to the prevailing one in a laissez-faire economy. Recall, however, from Proposition 2 that if the prior mean is high enough, equilibrium is unique and a small increase in the equilibrium cutoff increases welfare. Thus, if the prior mean is high enough (i.e. if $\kappa_2\mathcal{X}$ lies sufficiently to the right as is the case in Figure 3) there exists a value of $\mu^c > \mu^{LF}$ such that equilibrium remains unique and welfare is higher under cutoff μ^c than under cutoff μ^{LF}. Hence, even if the social planner anticipates that players will focus on cutoff $\tilde{\mu}$ instead of μ^{SP} in case $\tau = \tau'$, it is still optimal for him to tax investments. (Of course, the tax will have to be smaller than τ'.) Our main findings are summarized below.

Proposition 4. *If the prior mean is high enough, it is optimal to tax first-period investments. Furthermore, any socially optimal cutoff μ^{SP} can be implemented through an appropriate first-period investment tax τ and second-period investment tax rate of zero. Implementation, however, need not be unique.*

According to (perhaps recent) conventional wisdom, governments should not intervene in the presence of an investment bubble as one cannot ex ante know whether it is due to fundamentals (corresponding to the case in which $\theta > 0$ in our model) or whether it is the result of incorrect stories. Alan Greenspan's quote in our introduction, for example, nicely illustrates the wisdom that prevailed in 1999. Our model questions this rationale for non-intervention: Even if policymakers in contrast to market participants receive no private signal about the state of the world, the policymakers' knowledge of $\bar{\theta}$ can still be used to improve welfare.[22] In particular, in the presence of sufficiently favorable public information, investments should be taxed.

More broadly, Propositions 2 and 3 are consistent with the idea that investment policy should be countercyclical: when $\bar{\theta}$ is high (which is likely to occur when many players have invested in the previous period(s)), investments should be taxed, while if $\bar{\theta} \to -\infty$ (i.e. when expected investment activity is zero) investment should not be taxed.

7 Final Remarks

We analyzed some policy implications of social learning when players are fully rational and better informed than the policymaker. Our model is particularly useful when public information is conducive to investing—which typically happens during "boom times" or when many "stories" circulate praising the profit prospects of the investment opportunity.

[22] Greenspan was primarily worried about the existence of an investment boom in the U.S. stockmarket, i.e. in a context in which prices supposedly aggregate information. As our model is void of any price mechanism, one might argue that Greenspan's quote does not really apply to our set-up. We feel, however, that (perhaps until recently) the vast majority of policy-makers would agree (or would have agreed) with Greenspan even in a fixed-price context.

In this case, we establish that, in a laissez-faire economy, too many types are investing early and investments should therefore be taxed. To establish this result, we assumed that the social planner uses a time-varying tax/subsidy scheme. In general, one would not expect the government to frequently change investment policy on the basis of the latest investment activity. It is important to realize, however, that we only worked with a time-varying investment tax to simplify the analysis of the optimal cutoffs. In particular, we have been able to show that even if τ were fixed for two periods, it is still optimal to tax investments if the prior mean is sufficiently high. Intuitively, if the social planner raises the investment tax τ from zero to ϵ, she raise the first-period equilibrium cutoff which increases welfare. An increase in the investment tax τ, however, also distorts time-two investment decisions. This welfare loss, however, is a second-order effect. Furthermore, we have also been able to show that the issue of non-unique implementation disappears with permanent taxes. To be more specific, if the tax τ is set for two periods, and if the prior mean is sufficiently high, the optimal time-one cutoff μ^{SP} can be *uniquely* implemented by appropriately choosing τ.[23]

We haven chosen a two-player setup for our model. The general N player game is difficult to analyze.[24] One "simple" alternative, however, would be to consider a model with a continuum of players. In that variation, for any given symmetric equilibrium cutoff, social learning would be perfect and hence a laissez-faire policy optimal. To circumvent this unrealistic feature, one needs to assume social learning to be imperfect. One possibility is to assume observational noise as in Chamley (2004a) or Dasgupta (2007). In such a setup, Player i's distribution about the other players' posterior means (i.e. $f(\mu_j|\mu_i)$) would still be computed in the same way as in our two-player model. Therefore, if the prior mean is "very high" an inframarginal type expects—for "many" realizations of the state of the world—a large mass of players to invest at time one. As noisy observation of past investment behavior is then expected to reveal relatively little information about the realized state of the world, we conjecture that—as in our

[23] The proofs of these results are available upon request

[24] We have been able to establish, however, that equilibrium is unique with N players when the state of the world θ is drawn from a Laplacian distribution. Again, the proof is available upon request.

model—the inframarginal types prefer the social planner to raise the equilibrium cutoff via taxes. One drawback of such an approach, however, is that the observational noise is completely exogenously specified. An alternative assumption is that each player can only observe some (neighboring) players' first-period decision.[25]

In our model information can only be transmitted through actions. As there are no payoff externalities, it is natural to ask why information cannot be transmitted through words instead. If players can fully exchange their private information via cheap talk, an efficient equilibrium of course exists. We feel, however, that this simple argument is misleading as communication—even where allowed and feasible—is often imperfect. Suppose, for example, that player one is asked to reveal her type to the other player(s) prior to the waiting game. As her signal is imperfect, she also wants to learn the other player(s)' signal(s). She therefore has an incentive to send the message which maximizes her gain of waiting. In an analysis of cheap talk, Gossner and Melissas (2006) have shown that this game may—depending on the values of the parameters—be characterized by a unique monotone equilibrium in which all types send the same message, i.e. information can only be revealed through actions. More generally, we believe the study of waiting games in the presence of imperfect communication to be an interesting avenue for future research.

Another noteworthy aspect of our model is that investment costs are exogenous.[26] In many applications in which policymakers are concerned about investment bubbles—such as stock market or housing market bubbles—one would expect investment costs to increase in the number of present and past investments. In an exogenous queue model with a competitive market maker, Park and Sabourian (2010) establish that herding is possible even if markets are informationally efficient. An interesting question for future research is how these results extend to endogenous queue setting and whether it is also optimal to tax investments during boom times in such a model.

We assumed that players are fully rational. Eyster and Rabin (2009) nicely highlight

[25] Our model, for example, can be seen as a special case in which countably many players live on a circle and each player only observes her right-hand neighbor.

[26] See also the discussion in Footnote 22.

some counterintuitive features of the rational learning model in an exogenous queue environment and propose a plausible alternative learning model. An interesting question is whether the introduction of inferentially naive and/or cursed players strengthens or qualifies our "taxation during booms" result in an endogenous queue environment.

Appendix

Definitions and Preliminaries

Throughout the appendix F, f, h, and r represent, respectively, the c.d.f., the p.d.f., the hazard rate $\left(\equiv \frac{f(\cdot)}{1-F(\cdot)}\right)$, and the reverse hazard rate $\left(\equiv \frac{f(\cdot)}{F(\cdot)}\right)$ of the standard normal distribution. We will also use the following notations: $\alpha \equiv \frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\epsilon^2}$, $\beta \equiv \frac{\frac{2}{\sigma_\epsilon^2}}{\frac{1}{\sigma_\theta^2}+\frac{2}{\sigma_\epsilon^2}}$, $\sigma_p^2 \equiv \frac{\sigma_\theta^2\sigma_\epsilon^2}{\sigma_\theta^2+\sigma_\epsilon^2}$, $\sigma_2^2 \equiv \sigma_p^2+\sigma_\epsilon^2$, $\sigma_o^2 \equiv \alpha^2\sigma_2^2$, $\sigma_\mu^2 \equiv \alpha^2(\sigma_\theta^2+\sigma_\epsilon^2)$, $\kappa_1 \equiv \frac{1-\alpha}{\sigma_o}$, $\kappa_2 \equiv \frac{1}{2}\beta\sigma_2$, $x(\mu^c,\mu_i) \equiv \frac{\mu^c-\alpha\mu_i-(1-\alpha)\bar{\theta}}{\sigma_o}$, $\mathcal{X}(\eta) \equiv \frac{\delta f(\eta)}{1-\delta(1-F(\eta))}$, $g(\mu) \equiv \mu-\kappa_2\mathcal{X}(\kappa_1(\mu-\bar{\theta}))$, and $\phi(\mu) \equiv \mu+\kappa_2 h(x(\mu^*,\mu))$.

In our set-up (see DeGroot (1984) for proofs) $\theta|s_i \sim N(\mu_i,\sigma_p^2)$, where

$$\mu_i = \alpha s_i + (1-\alpha)\bar{\theta}. \tag{19}$$

As ϵ_j is independent from θ and ϵ_i, $s_j|s_i = \theta|s_i+\epsilon_j$. As $\epsilon_j \sim N(0,\sigma_\epsilon^2)$, $s_j|s_i \sim N(\mu_i,\sigma_p^2+\sigma_\epsilon^2)$. Furthermore, $\mu_j = \alpha s_j + (1-\alpha)\bar{\theta}$, and thus,

$$\mu_j|s_i \sim N(\alpha\mu_i+(1-\alpha)\bar{\theta},\sigma_o^2). \tag{20}$$

Hence,

$$\Pr(\mu_j > \mu^*|\mu_i) = 1-F\left(\frac{\mu^*-\alpha\mu_i-(1-\alpha)\bar{\theta}}{\sigma_o}\right), \tag{21}$$

and

$$\Pr(\mu_j > \mu^*|\mu_i=\mu^*) = 1-F\left(\kappa_1\left(\mu^*-\bar{\theta}\right)\right).$$

Lemma 2. *If signals and the state of the world are drawn from Normal distributions,*

1. *$E(\theta|\mu_i,\mu_j > \mu^c)$ and $E(\theta|\mu_i,\mu_j < \mu^c)$ are increasing in μ_i and μ^c.*

2. *One has:*

$$E(\theta|\mu_i, \mu_j > \mu^c) = \mu_i + \kappa_2 h(x(\mu^c, \mu_i)) \text{ and } E(\theta|\mu_i, \mu_j < \mu^c) = \mu_i - \kappa_2 r(x(\mu^c, \mu_i)).$$

3. *For any first-period cutoff μ^c, there exist unique-second period cutoffs $\underline{\mu}^0$ and $\underline{\mu}^1$ such that $E(\theta|\mu_i = \underline{\mu}^0, \mu_j < \mu^c) = 0$ and $E(\theta|\mu_i = \underline{\mu}^1, \mu_j > \mu^c) = 0$.*

4. *There exists a unique $\underline{\mu} < 0$ such that $E(\theta|\underline{\mu}, \mu_j > \underline{\mu}) = 0$. There also exists a unique $\bar{\mu} > 0$ such that $E(\theta|\bar{\mu}, \mu_j < \bar{\mu}) = 0$. If $\mu^c < \underline{\mu}$, no active type invests at time two. If $\mu^c > \bar{\mu}$, some active types invest at time two even if no one did so at time one. If $\mu^c \in (\underline{\mu}, \bar{\mu})$, an active player only invests at time two if she received good news.*

Proof: We first prove points 1 and 2 of the lemma. A well known statistical result (see DeGroot (1984) for a proof) is that if $\theta \sim N(\bar{\theta}, \sigma_\theta^2)$ and if $\epsilon_i \sim N(0, \sigma_\epsilon^2)$, then $\theta|s_i, s_j$ also tends to a normal and

$$E(\theta|s_i, s_j) = \beta \frac{s_i + s_j}{2} + (1-\beta)\bar{\theta}.$$

We first tackle the case in which $\mu_j > \mu^c$. It follows from 19 that $\mu_j > \mu^c \Leftrightarrow s_j > s^c \equiv \frac{\mu^c - (1-\alpha)\bar{\theta}}{\alpha}$. One has,

$$\begin{aligned} E(\theta|\mu_i, \mu_j > \mu^c) &= \int \left[\beta \frac{s_i + s_j}{2} + (1-\beta)\bar{\theta} \right] f(s_j|s_i, s_j \geq s^c) ds_j, \\ &= \frac{\beta}{2} s_i + \frac{\beta}{2} E(s_j|s_i, s_j > s^c) + (1-\beta)\bar{\theta}. \end{aligned} \tag{22}$$

From the explanations provided after 19, we know that $s_j|s_i, s_j > s^c$ is a left-truncated normal distribution with mean μ_i, variance σ_2^2 and truncation point s^c. Using Johnson et al. (1995) to calculate the expectation of a truncated normal variable, one has

$$E(s_j|s_i, s_j > s^c) = \mu_i + h\left(\frac{s^c - \mu_i}{\sigma_2}\right)\sigma_2. \tag{23}$$

Replacing s^c by $\frac{\mu^c - (1-\alpha)\bar{\theta}}{\alpha}$ and taking into account that $\sigma_o^2 = \alpha^2(\sigma_p^2 + \sigma_\epsilon^2) = \alpha^2\sigma_2^2$, allow us to rewrite 23 as $E(s_j|s_i, s_j > s^c) = \mu_i + h(x(\mu^c, \mu_i))\sigma_2$. Inserting this last equality into 22, and taking into account the fact that $\mu_i = \alpha s_i + (1-\alpha)\bar{\theta}$ and that $\beta(1+\alpha) = 2\alpha$, yields:

$$E(\theta|\mu_i, \mu_j > \mu^c) = \mu_i + \kappa_2 h(x(\mu^c, \mu_i)). \tag{24}$$

Differentiating 24, and taking into account that $\frac{\alpha}{\sigma_o}\kappa_2 = \frac{1}{2}\beta$, one has

$$\frac{\partial E(\theta|\mu_i, \mu_j > \mu^c)}{\partial \mu_i} = 1 - \frac{1}{2}\beta h'(x(\mu^c, \mu_i)). \tag{25}$$

As is well known (see, e.g. Greene (1993), Theorem 22.2), the slope of the hazard rate of a standard normal distribution, $h'(z) \in (0,1)\ \forall z$. This insight, combined with the fact that $\beta \in [0,1]$, allows us to conclude that $\frac{\partial E(\theta|\mu_i, \mu_j > \mu^c)}{\partial \mu_i} > 0$. Differentiating 24 with respect to μ^c and taking into account that $\frac{\kappa_2}{\sigma_o} = \frac{\beta}{2\alpha}$, one has

$$\frac{\partial E(\theta|\mu_i, \mu_j > \mu^c)}{\partial \mu^c} = \frac{\beta}{2\alpha} h'(x(\mu^c, \mu_i)).$$

As $h'(z) \in (0,1)$ and as both α and β are positive, we conclude that $\frac{\partial E(\theta|\mu_i, \mu_j > \mu^c)}{\partial \mu^c} > 0$.

We now tackle the case in which $\mu_j < \mu^c$. As above,

$$E(\theta|\mu_i, \mu_j < \mu^c) = \frac{\beta}{2} s_i + \frac{\beta}{2} E(s_j|\mu_i, \mu_j < \mu^c) + (1-\beta)\bar{\theta}. \tag{26}$$

From Johnson et al (1995), we know that

$$E(s_j|\mu_i, s_j < s^c) = \mu_i - r\left(\frac{s^c - \mu_i}{\sigma_2}\right)\sigma_2. \tag{27}$$

Inserting 27 into 26, replacing s^c by $\frac{\mu^c - (1-\alpha)\bar{\theta}}{\alpha}$, and taking into account that $\frac{\beta}{2\alpha} = \frac{1}{1+\alpha}$ and that $1 - \beta = \frac{1-\alpha}{1+\alpha}$, yields:

$$E(\theta|\mu_i, \mu_j < \mu^c) = \mu_i - \kappa_2 r(x(\mu^c, \mu_i)). \tag{28}$$

Differentiating this last equation, and using the fact that $\frac{\alpha}{\sigma_o}\kappa_2 = \frac{1}{2}\beta$, one has

$$\frac{\partial E(\theta|\mu_i, \mu_j < \mu^c)}{\partial \mu_i} = 1 - \frac{1}{2}\beta r'(x(\mu^c, \mu_i)).$$

It is well known (see, e.g. Greene (1993), Theorem 22.2) that $r'(\cdot) \in (-1, 0)$. As $\beta \in [0,1]$, we conclude that $\frac{\partial E(\theta|\mu_i, \mu_j < \mu^c)}{\partial \mu_i}$ is positive. Differentiating 28 with respect to μ^c, one has $\frac{\partial E(\theta|\mu_i, \mu_j < \mu^c)}{\partial \mu^c} = -\frac{\kappa_2}{\sigma_o} r'(\cdot)$, which is positive as $r'(\cdot) < 0$.

We now prove point 3 of the lemma. Recall that $\phi(\mu) \equiv \mu + \kappa_2 h(x(\mu^c, \mu))$. At the second-period cutoff $E(\theta|\underline{\mu}^1, \mu_j > \mu^c) = 0$, which is equivalent to $\phi(\underline{\mu}^1) = 0$. It follows from 25 that $\phi(\mu)$ is increasing in μ. Hence, if there exists a solution, it is unique. We are left to establish that a solution exists. First, observe that $\lim_{\mu\to\infty} \phi(\mu) > 0$. Second, note that $\lim_{\mu\to-\infty} \left[\mu + \kappa_2 h\left(x(\mu^c, \mu)\right)\right] < 0$, is equivalent to $\lim_{\mu\to-\infty} \left[\frac{\mu+\kappa_2 h(x(\mu^c,\mu))}{\mu}\right] > 0$, which by l'Hôpital's rule is equivalent to $\lim_{\mu\to-\infty} \left[1 - \frac{\beta}{2} h'\left(x(\mu^c, \mu)\right)\right] > 0$. Since $h' \in (0, 1)$ and $\beta < 1$ this holds, which establishes the existence of $\underline{\mu}^1$.

Let $\psi(\mu) \equiv \mu - \kappa_2 r(x(\mu^c, \mu))$, and observe that $\underline{\mu}^0$ is implicitly defined as $\psi(\underline{\mu}^0) = 0$. Using the fact that $\frac{\alpha\kappa_2}{\sigma_o} = \frac{1}{2}\beta$, one has $\psi'(\mu) = 1 + \frac{1}{2}\beta r'(x(\mu^c, \mu)) > 0$, where the inequality follows from the fact that $r'(\cdot) \in (-1, 0)$. Hence, if a solution exists, it is unique. Note that $\lim_{\mu\to-\infty} \psi(\mu) < 0$. Using l'Hôpital's rule, $\lim_{\mu\to\infty} \left[\mu - \kappa_2 r(x(\mu^c, \mu))\right] > 0$, is equivalent to $\lim_{\mu\to\infty} 1 + \frac{1}{2}\beta r'(x(\mu^c, \mu)) > 0$, which is satisfied as $r'(z) \in (-1, 0)$. Hence, a solution exists.

We now prove point 4 of the lemma. Let $\hat{\psi}(\mu) \equiv \mu - \kappa_2 r(\kappa_1(\mu - \bar{\theta}))$, and observe that $\bar{\mu}$ is implicitly defined as $\hat{\psi}(\bar{\mu}) = 0$. Using the fact that $\kappa_1\kappa_2 = \frac{1-\alpha}{1+\alpha}$, yields $\frac{\partial\hat{\psi}}{\partial\mu} = 1 - \frac{1-\alpha}{1+\alpha} r'(\cdot) > 0$, as $r'(\cdot) \in (-1, 0)$. This insight, combined with the fact that $\lim_{\mu\to-\infty} \hat{\psi}(\mu) = -\infty$ and that $\lim_{\mu\to\infty} \hat{\psi}(\mu) = \infty$ proves the existence of a unique $\bar{\mu}$. In the paper, we prove that $0 < \bar{\mu}$. Suppose $\bar{\mu} < \mu^c \leq \mu_i$. Then $0 \equiv E(\theta|\bar{\mu}, \mu_j < \bar{\mu}) < E(\theta|\bar{\mu}, \mu_j < \mu^c) \leq E(\theta|\mu_i, \mu_j < \mu^c)$, where all inequalities follow from Point 1 of this lemma. Hence, if $\bar{\mu} < \mu^c$, some active types will invest at time two even if no one invested at time one. Similarly, let $\hat{\phi}(\mu) \equiv \mu + \kappa_2 h(\kappa_1(\mu - \bar{\theta}))$, and observe that $\underline{\mu}$ is implicitly defined as $\hat{\phi}(\underline{\mu}) = 0$. Using the fact that $\kappa_1\kappa_2 = \frac{1-\alpha}{1+\alpha}$, yields $\frac{\partial\hat{\phi}}{\partial\mu} = 1 + \frac{1-\alpha}{1+\alpha} h'(\cdot) > 0$. This insight, combined with the fact that $\lim_{\mu\to-\infty} \hat{\phi}(\mu) = -\infty$ and that $\lim_{\mu\to\infty} \hat{\phi}(\mu) = \infty$ proves the existence of a unique $\underline{\mu}$. In the paper, we prove that $\underline{\mu} < 0$. Suppose $\mu_i \leq \mu^c < \underline{\mu}$. Then $E(\theta|\mu_i, \mu_j < \mu^c) < E(\theta|\mu_i, \mu_j > \mu^c) < E(\theta|\underline{\mu}, \mu_j > \underline{\mu}) = 0$, where the first inequality follows from the fact that observing $\mu_j > \mu^c$ is good news and where the second inequality follows from Point 1 of this lemma. Hence, if $\mu^c < \underline{\mu}$, no active type invests at time two. Finally, suppose that $\mu^c \in (\underline{\mu}, \bar{\mu})$. As $\underline{\mu} < \mu^c$, $0 = E(\theta|\underline{\mu}, \mu_j > \underline{\mu}) < E(\theta|\mu^c, \mu_j > \mu^c)$, where the inequality follows from point 1 of this lemma. By continuity, there exist values of μ_i close to (but less than) μ^c such that Player i wants to invest upon getting good news. As $\mu^c < \bar{\mu}$, it follows from point 1 of this lemma that $E(\theta|\mu^c, \mu_j < \mu^c) < E(\theta|\bar{\mu}, \mu_j < \bar{\mu}) = 0$. Hence, no active type invests at time two if no one did so at time one.■

Proof of Lemma 1

Observe that for any finite μ_1 and μ_2^c, $E(\theta|\mu_1, \mu_2 < \mu_2^c) < E(\theta|\mu_1, \mu_2 > \mu_2^c)$. There are thus three possibilities:

(i) $E(\theta|\mu_1, \mu_2 < \mu_2^c) < E(\theta|\mu_1, \mu_2 > \mu_2^c) \leq 0$,

(ii) $E(\theta|\mu_1, \mu_2 < \mu_2^c) \leq 0 < E(\theta|\mu_1, \mu_2 > \mu_2^c)$, and

(iii) $0 < E(\theta|\mu_1, \mu_2 < \mu_2^c) < E(\theta|\mu_1, \mu_2 > \mu_2^c)$.

In case (i), $\Delta(\cdot) = \mu_1 - \tau$, which is increasing in μ_1.

In case (ii), $\Delta(\cdot) = \mu_1 - \tau - \delta \Pr(\mu_2 > \mu_2^c|\mu_1) E(\theta|\mu_1, \mu_2 > \mu_2^c)$. Observe that

$$\mu_1 = \Pr(\mu_2 > \mu_2^c|\mu_1) E(\theta|\mu_1, \mu_2 > \mu_2^c) + \Pr(\mu_2 < \mu_2^c|\mu_1) E(\theta|\mu_1, \mu_2 < \mu_2^c).$$

Inserting this last equality into $\Delta(\cdot)$, yields

$$\Delta(\cdot) = (1-\delta) \Pr(\mu_2 > \mu_2^c|\mu_1) E(\theta|\mu_1, \mu_2 > \mu_2^c) + \Pr(\mu_2 < \mu_2^c|\mu_1) E(\theta|\mu_1, \mu_2 < \mu_2^c) - \tau. \tag{29}$$

Differentiating this last expression of $\Delta(\cdot)$ yields:

$$\begin{aligned} \frac{\partial \Delta(\mu_1, \mu_2^c)}{\partial \mu_1} &= (1-\delta) \frac{\partial \Pr(\mu_2 > \mu_2^c|\mu_1)}{\partial \mu_1} E(\theta|\mu_1, \mu_2 > \mu_2^c) \\ &+ (1-\delta) \frac{\partial E(\theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_1} \Pr(\mu_2 > \mu_2^c|\mu_1) \\ &+ \frac{\partial \Pr(\mu_2 < \mu_2^c|\mu_1)}{\partial \mu_1} E(\theta|\mu_1, \mu_2 < \mu_2^c) \\ &+ \frac{\partial E(\theta|\mu_1, \mu_2 < \mu_2^c)}{\partial \mu_1} \Pr(\mu_2 < \mu_2^c|\mu_1). \end{aligned} \tag{30}$$

In case (ii), $E(\theta|\mu_1, \mu_2 > \mu_2^c) > 0$. As $\frac{\partial \Pr(\mu_2 > \mu_2^c|\mu_1)}{\partial \mu_1}$ is also positive, the first term of the RHS of 30 is positive. Moreover, from Lemma 2 we know that both $\frac{\partial E(\theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_1}$ and $\frac{\partial E(\theta|\mu_1, \mu_2 < \mu_2^c)}{\partial \mu_1}$ are positive. Hence, the second and the fourth term of the RHS of 30 are also positive. In case (ii), $E(\theta|\mu_1, \mu_2 < \mu_2^c) \leq 0$. This assumption, combined with the fact that $\frac{\partial \Pr(\mu_2 < \mu_2^c|\mu_1)}{\partial \mu_1} < 0$, proves that the third term of the RHS of 30 is also positive. Finally, in case (iii) $\Delta(\cdot) = (1-\delta)\mu_1 - \tau$, which is also increasing in μ_1.■

Proof of Proposition 1

Recall that $\kappa_1 = \frac{1-\alpha}{\sigma_o}$, $\kappa_2 = \frac{1}{2}\beta\sigma_2$, $\sigma_2 = \sqrt{\sigma_p^2 + \sigma_\epsilon^2}$ and that $x(\mu_2^c, \mu_1) = \frac{\mu_2^c - \alpha\mu_1 - (1-\alpha)\bar{\theta}}{\sigma_o}$. Recall that

$$\mathcal{X}(\eta) = \frac{\delta f(\eta)}{1 - \delta(1 - F(\eta))}. \tag{31}$$

LEMMA 3. *There exists a unique $\hat{\eta} < 0$ such that $\mathcal{X}(\hat{\eta}) = -\hat{\eta}$. $\mathcal{X}(\eta)$ increases until $\eta = \hat{\eta}$, after which it decreases.* $\lim_{\eta\to-\infty} \mathcal{X}(\eta) = \lim_{\eta\to+\infty} \mathcal{X}(\eta) = 0$ *and* $\lim_{\eta\to-\infty} \mathcal{X}'(\eta) = \lim_{\eta\to+\infty} \mathcal{X}'(\eta) = 0$. *$\mathcal{X}''(\eta) > 0$ if $\eta < \eta^m$ (where $\eta^m < \hat{\eta}$) and $\mathcal{X}''(\eta) < 0$ if $\eta \in (\eta^m, \hat{\eta})$.* $\lim_{\delta\to1} \hat{\eta} = -\infty$. $\lim_{\delta\to0} \mathcal{X}'(\eta^m) = 0$ *and* $\lim_{\delta\to1} \mathcal{X}'(\eta^m) = \infty$.

Proof: Observe that $\mathcal{X}(\eta) > 0$ for $\delta > 0$. Hence, $\mathcal{X}(\eta) > -\eta$, $\forall\eta > 0$. Mere introspection of 31 reveals that for sufficiently low values of η, $\mathcal{X}(\eta) < -\eta$. By continuity, there exists at least one $\hat{\eta} < 0$ such that $\mathcal{X}(\hat{\eta}) = -\hat{\eta}$. Observe that the right hand side of the equality decreases in η and that

$$\frac{\partial\mathcal{X}(\eta)}{\partial\eta} = \mathcal{X}'(\eta) = -\mathcal{X}(\eta)[\eta + \mathcal{X}(\eta)]. \tag{32}$$

This slope is equal to zero if and only if $\mathcal{X}(\eta) = -\eta$. Hence, whenever $\mathcal{X}(\eta) = -\eta$, the right hand side of the equality strictly decreases in η, while its left hand side remains constant. As the slope of $\mathcal{X}(\eta)$ varies smoothly with changes in η, this implies that there is exactly one $\hat{\eta} < 0$ such that $\mathcal{X}(\hat{\eta}) = -\hat{\eta}$.

Note that if $\eta < \hat{\eta}$, $\mathcal{X}(\eta) < -\eta$, and $\mathcal{X}'(\eta) > 0$. Similarly, if $\eta > \hat{\eta}$, $\mathcal{X}'(\eta) < 0$. As the denominator of 31 is greater than $1 - \delta$ and as $\lim_{\eta\to+\infty} f(\eta) = \lim_{\eta\to-\infty} f(\eta) = 0$, one has: $\lim_{\eta\to-\infty} \mathcal{X}(\eta) = \lim_{\eta\to+\infty} \mathcal{X}(\eta) = 0$.

On the basis of 32, one has

$$\lim_{\eta\to\infty} \mathcal{X}'(\eta) = \lim_{\eta\to\infty} \mathcal{X}(\eta)(-\eta) - \lim_{\eta\to\infty} [\mathcal{X}(\eta)]^2.$$

Since $\lim_{\eta\to\infty} \mathcal{X}(\eta) = 0$, $\lim_{\eta\to\infty}[\mathcal{X}(\eta)]^2 = 0$. Observe also that

$$\mathcal{X}(\eta)(-\eta) = \frac{\delta f(\eta)(-\eta)}{1 - \delta(1 - F(\eta))} = \frac{\delta f'(\eta)}{1 - \delta(1 - F(\eta))}.$$

As $\lim_{\eta\to\infty} f'(\eta) = 0$ and as $\delta < 1$, $\lim_{\eta\to\infty} \mathcal{X}(\eta)(-\eta) = 0$. Hence, $\lim_{\eta\to\infty} \mathcal{X}'(\eta) = 0$. By the same reasoning, $\lim_{\eta\to-\infty} \mathcal{X}'(\eta)$ is zero.

Observe that

$$\mathcal{X}''(\eta) = -\eta\mathcal{X}'(\eta) - 2\mathcal{X}'(\eta)\mathcal{X}(\eta) - \mathcal{X}(\eta). \tag{33}$$

As $\lim_{\eta\to-\infty}\mathcal{X}(\eta) = \lim_{\eta\to-\infty}\mathcal{X}'(\eta) = 0$,

$$\lim_{\eta\to-\infty}\mathcal{X}''(\eta) = \lim_{\eta\to-\infty} -\eta\mathcal{X}'(\eta) \geq 0,$$

and for η sufficiently small $\mathcal{X}''(\eta) > 0$. As $\mathcal{X}'(\hat{\eta}) = 0$, it follows from 33 that $\mathcal{X}''(\hat{\eta}) < 0$. By continuity, there exists at least one $\eta^m \in (-\infty, \hat{\eta})$ such that $\mathcal{X}''(\eta^m) = 0$. Differentiating 33, and evaluating at the point $\eta = \eta^m$, one has

$$\mathcal{X}'''(\eta)\big|_{\eta=\eta^m} = -\mathcal{X}'(\eta^m) - 2\left(\mathcal{X}'(\eta^m)\right)^2 < 0,$$

where the inequality follows from the fact that $\mathcal{X}'(\eta^m) > 0$, as $\eta^m < \hat{\eta}$. We conclude that η^m is unique.

Recall that $\hat{\eta} < 0$. Suppose $\lim_{\delta\to1}\mathcal{X}'(\hat{\eta}) = 0$ for some $\hat{\eta} \in (-\infty, 0)$. It follows from 32 that

$$\lim_{\delta\to1}\mathcal{X}'(\hat{\eta}) = 0 \Leftrightarrow -\hat{\eta} = \lim_{\delta\to1}\mathcal{X}(\hat{\eta}) = \frac{f(\hat{\eta})}{F(\hat{\eta})} = r(\hat{\eta}).$$

It is easy to check that $\frac{\partial r(\eta)}{\partial\eta} = -r(\eta)(r(\eta) + \eta)$. Hence, $r'(\hat{\eta}) = 0$. This, however, contradicts the fact that $\forall\eta \in (-\infty, \infty)$, $r'(\eta) < 0$ (see Greene, 1993, Theorem 22.2). Thus, $\lim_{\delta\to1}\hat{\eta} = -\infty$.

Observe that $\lim_{\delta\to0}\mathcal{X}(\eta) = 0\ \forall\eta$. Hence, $\lim_{\delta\to0}\mathcal{X}'(\eta^m) = 0$.

As $\eta^m < \hat{\eta}$ and $\lim_{\delta\to1}\hat{\eta} = -\infty$, $\lim_{\delta\to1}\eta^m = -\infty$. Therefore, $\lim_{\delta\to1}\mathcal{X}(\eta^m) = \frac{f(-\infty)}{F(-\infty)} = \infty$, where the last equality follows from l'Hôpital's rule. It follows from 33 that

$$\mathcal{X}''(\eta^m) = 0 \Leftrightarrow \eta^m = -\mathcal{X}(\eta^m)\left(\frac{1}{\mathcal{X}'(\eta^m)} + 2\right). \tag{34}$$

Recall that $\mathcal{X}'(\eta^m) = -\mathcal{X}(\eta^m)[\eta^m + \mathcal{X}(\eta^m)]$. Replacing η^m on the right-hand side of this equality by the right-hand side of the last equality in 34, and rearranging, one has

$$\frac{[\mathcal{X}'(\eta^m)]^2}{1 + \mathcal{X}'(\eta^m)} = [\mathcal{X}(\eta^m)]^2. \tag{35}$$

As $\lim_{\delta\to1}\mathcal{X}(\eta^m) = \infty$, $\lim_{\delta\to1}[\mathcal{X}(\eta^m)]^2 = \infty$. Thus $\lim_{\delta\to1}\frac{[\mathcal{X}'(\eta^m)]^2}{1+\mathcal{X}'(\eta^m)} = \infty$, which implies that $\lim_{\delta\to1}\mathcal{X}'(\eta^m) = \infty$. ■

Call LHS (RHS) the left-hand side (respectively right-hand side) of equation 5 after replacing μ^* by μ, and observe that

$$\frac{\partial LHS}{\partial \mu} = 1 \text{ and that } \frac{\partial RHS}{\partial \mu} = \kappa_1 \kappa_2 \mathcal{X}'(\kappa_1(\mu - \bar{\theta})) = \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\kappa_1(\mu - \bar{\theta})),$$

where the last equality follows from the fact that $\frac{\beta}{2\alpha} = \frac{1}{1+\alpha}$. From Lemma 3 we know that $\mathcal{X}'(\kappa_1(\mu^* - \bar{\theta}))$ is maximal when $\kappa_1(\mu^* - \bar{\theta}) = \eta^m$.[27] As $\mathcal{X}''(\cdot) > 0$ when $\kappa_1(\mu^* - \bar{\theta}) < \eta^m$, as $\mathcal{X}''(\cdot) < 0$ when $\kappa_1(\mu^* - \bar{\theta}) \in (\eta^m, \hat{\eta})$ and as $\mathcal{X}'(\cdot) < 0$ when $\kappa_1(\mu^* - \bar{\theta}) > \hat{\eta}$, it follows that $\forall \bar{\theta}$, there exists a unique equilibrium in symmetric switching strategies if and only if

$$\left.\frac{\partial RHS}{\partial \mu}\right|_{\mu=\mu^*=\frac{\eta^m}{\kappa_1}+\bar{\theta}} \leq \left.\frac{\partial LHS}{\partial \mu}\right|_{\mu=\mu^*=\frac{\eta^m}{\kappa_1}+\bar{\theta}} \Leftrightarrow \mathcal{X}'(\eta^m) \leq \frac{1+\alpha}{1-\alpha}.$$

We know from Lemma 3 that $\lim_{\eta^m \to \infty} \mathcal{X}'(\eta^m) = \lim_{\eta^m \to -\infty} \mathcal{X}'(\eta^m) = 0$. Moreover, mere observation of 31 also reveals that $\mathcal{X}(\eta^m) < \infty$ if $\delta < 1$. Hence, $\mathcal{X}'(\eta^m) = -\mathcal{X}(\eta^m)(\eta^m + \mathcal{X}(\eta^m))$ is finite whenever $\delta < 1$. Observe that $\lim_{\sigma_\theta^2 \to \infty} \alpha = \lim_{\sigma_\epsilon^2 \to 0} \alpha = 1$, which, combined with our finding that $\mathcal{X}'(\eta^m)$ is finite, implies that $\lim_{\sigma_\theta^2 \to \infty} \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m)$ $= \lim_{\sigma_\epsilon^2 \to 0} \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m) = 0$. By continuity, there exists a $(\sigma_\theta^2)^c < \infty$ ($(\sigma_\epsilon^2)^c > 0$) such that $\forall \sigma_\theta^2 > (\sigma_\theta^2)^c$ ($\forall \sigma_\epsilon^2 < (\sigma_\epsilon^2)^c$), $\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m) \leq 1$. This establishes claims (1) and (2) of the proposition.

Since $\mathcal{X}(\eta^m) = 0$ when $\delta = 0$, one has $\left.\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m)\right|_{\delta=0} = 0$. By continuity, there exists a $\delta^c \in (0,1]$ such that $\forall \delta \leq \delta^c$, $\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m) \leq 1$. This establishes claim (3) of the proposition.

Recall that

$$g(\mu) = \mu - \kappa_2 \mathcal{X}(\kappa_1(\mu - \bar{\theta})), \tag{36}$$

and observe that equilibrium condition 5 is equivalent to $g(\mu^*) = 0$. If $\mu < 0$, $g(\mu) < 0$. Thus, $\mu^* > 0$. Hence, if $\bar{\theta} \leq 0$, $\kappa_1(\mu^* - \bar{\theta}) > 0$. It then follows from Lemma 3 that $\mathcal{X}'(\kappa_1(\mu^* - \bar{\theta})) < 0$. This establishes claim (4) of the proposition.

Suppose that if Player i waits, she perfectly learns the state of the world, which gives an upper bound on the value of learning. Player i's gain of waiting then equals $\delta \Pr(\theta >$

[27] As a unit increase in $\bar{\theta}$ leads to a translation of $\mathcal{X}(\cdot)$ to the right by one unit (as shown in Figure 1), it follows that there exists a unique $\bar{\theta}$ such that $\kappa_1(\mu^* - \bar{\theta}) = \eta^m$.

$0|\mu_i)E(\theta|\mu_i, \theta > 0)$. Observe that for high enough a μ_i, $E(\theta|\mu_i, \theta > 0) \approx E(\theta|\mu_i) = \mu_i$. As $\delta < 1$, there exists a $\bar{\mu} < \infty$ such that $\bar{\mu} = \delta \Pr(\theta > 0|\bar{\mu})E(\theta|\bar{\mu}, \theta > 0)$. If $\mu > \bar{\mu}$ Player i strictly prefers to invest at time one. Hence, $\mu^* < \bar{\mu} < \infty$. As $\mu^* \in (0, \bar{\mu})$, $\kappa_1(\mu^* - \bar{\theta}) \to -\infty$, as $\bar{\theta} \to \infty$. It then follows from Lemma 3 that $\lim_{\bar{\theta}\to\infty} \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\kappa_1(\mu^* - \bar{\theta})) = 0$. By continuity, there exists a $\bar{\theta}_u$ such that if $\bar{\theta} \geq \bar{\theta}_u$, $\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\kappa_1(\mu^* - \bar{\theta})) \leq 1$. This establishes claim (5) of the proposition.

Recall from Lemma 3 that $\lim_{\delta\to 1} \hat{\eta} = -\infty$ and that $\mathcal{X}'(\eta) < 0$ when $\eta > \hat{\eta}$. Thus, if δ is close to one, μ cuts $\kappa_2\mathcal{X}(\kappa_1(\mu - \bar{\theta}))$ when $\mathcal{X}'(\cdot) < 0$, in which case equilibrium is unique. By continuity, there exists a $\bar{\delta} < 1$ such that $\mathcal{X}'(\kappa_1(\mu^* - \bar{\theta})) \leq 0$ for all $\delta \geq \bar{\delta}$. This establishes claim (6) of the proposition.■

Proof of the equivalence between Equations 7 and 8.

In this proof, k denotes the p.d.f. of some random variable. For example, $k(\theta) = f(\frac{\theta - \bar{\theta}}{\sigma_\theta})$. k obviously depends on the studied random variable. For example, it follows from our section "Definitions and Preliminaries" that $\theta \sim N(\bar{\theta}, \sigma_\theta^2)$ and that $\mu_i|\theta \sim N(\alpha\theta + (1-\alpha)\bar{\theta}, \sigma_\mu^2)$. Hence, $k(\theta) \neq k(\mu_i|\theta)$. In that sense, it would be more precise to use the notation k^θ and $k^{\mu_i|\theta}$ to respectively denote the p.d.f.'s of θ and $\mu_i|\theta$. In this proof, however, we avoid this cumbersome notation. This should not cause confusion. Observe that Equation 7 can be rewritten as

$$\begin{aligned} \frac{1}{2}W &= \int \Pr(\mu_i > \mu^c|\theta)\theta k(\theta)d\theta \\ &+ \delta \int \Pr\left(\mu_j > \mu^c, \mu_i \in \left[\min\{\underline{\mu}, \mu^c\}, \mu^c\right] \Big| \theta\right) \theta k(\theta) d\theta \\ &+ \delta \int \Pr\left(\mu_j < \mu^c, \mu_i \in \left[\min\{\underline{\mu}^0, \mu^c\}, \mu^c\right] \Big| \theta\right) \theta k(\theta) d\theta. \end{aligned} \tag{37}$$

Observe also that

$$\int \Pr(\mu_i > \mu^c|\theta)\theta k(\theta)d\theta = \int \int_{\mu^c}^{\infty} \frac{k(\mu_i, \theta)}{k(\theta)} d\mu_i \theta k(\theta) d\theta = \int_{\mu^c}^{\infty} \int \theta k(\theta|\mu_i) d\theta k(\mu_i) d\mu_i.$$

Trivially, $\mu_i = \int \theta k(\theta|\mu_i)d\theta$. Hence, the first integral of 37 is equal to $\int_{\mu^c}^{\infty} \mu_i k(\mu_i) d\mu_i$.

The second integral of 37 can be rewritten as

$$\int\int_{\mu^c}^{\infty}\int_{\min\{\underline{\mu},\mu^c\}}^{\mu^c}\frac{k(\mu_j,\mu_i,\theta)}{k(\theta)}d\mu_i d\mu_j \theta k(\theta)d\theta.$$

Changing the order of integration, the above integral can be rewritten as

$$\int_{\min\{\underline{\mu},\mu^c\}}^{\mu^c}\underbrace{\int_{\mu^c}^{\infty}\underbrace{\int\theta k(\theta|\mu_i,\mu_j)d\theta}_{E(\theta|\mu_i,\mu_j)}\,k(\mu_j|\mu_i)d\mu_j}_{\Pr(\mu_j>\mu^c|\mu_i)E(\theta|\mu_i,\mu_j>\mu^c)}\,k(\mu_i)d\mu_i.$$

Hence, the second integral of 37 is equal to $\int_{\min\{\underline{\mu},\mu^c\}}^{\mu^c}\Pr(\mu_j>\mu^c|\mu_i)E(\theta|\mu_i,\mu_j>\mu^c)k(\mu_i)d\mu_i$.

Using an identical procedure, the third integral of 37 can be rewritten as $\int_{\min\{\underline{\mu}^0,\mu^c\}}^{\mu^c}\Pr(\mu_j<\mu^c|\mu_i)E(\theta|\mu_i,\mu_j<\mu^c)k(\mu_i)d\mu_i$.■

Proof of Proposition 2.

The proof of this proposition is almost entirely explained in the body of the text. We are left to prove Inequality 12. It follows from 21 and from Lemma 2 that

$$\Pr(\mu_j>\mu^c|\mu_i)E(\theta|\mu_i,\mu_j>\mu^c)=\Big[1-F(x(\mu^c,\mu_i))\Big]\Big[\mu_i+\kappa_2 h(x(\mu^c,\mu_i))\Big]$$

Recall that $(1-F(z))h(z)=f(z)$, that $f'(z)=-f(z)z$ and that $x(\mu^c,\mu_i)=\frac{\mu^c-\alpha\mu_i-(1-\alpha)\bar{\theta}}{\sigma_o}$. Hence, the derivative of the right-hand side with respect to μ^c equals:

$$\frac{1}{\sigma_o}f(x(\mu^c,\mu_i))\left[-\mu_i+\kappa_2\frac{(1-\alpha)\bar{\theta}+\alpha\mu_i-\mu^c}{\sigma_o}\right],$$

which is positive if and only if the term between square brackets is. It is straightforward to show that $\frac{\kappa_2}{\sigma_o}=\frac{1}{1+\alpha}$. This insight permits us to conclude that the term between square brackets is positive if and only if $(1-\alpha)\bar{\theta}-\mu_i>\mu^c$.

Proof of Proposition 3.

Let $\tilde{\mu}\equiv(1-\alpha)\bar{\theta}-\mu^{LF}$ and recall that $x(\mu^{LF},\mu_i)=\frac{\mu^{LF}-\alpha\mu_i-(1-\alpha)\bar{\theta}}{\sigma_o}$.

LEMMA 4. *One has:* $\lim_{\bar{\theta}\to-\infty}\mu^{LF}=\lim_{\bar{\theta}\to-\infty}(1-\alpha)\bar{\theta}-\underline{\mu}^1$.

Proof: Rewriting $x(\mu^{LF}, \tilde{\mu})$ using $\kappa_2 = \frac{1}{2}\beta\sigma_2$ and $\frac{\beta}{2\alpha} = \frac{1}{1+\alpha}$ verifies that $\tilde{\mu} = -\kappa_2 x(\mu^{LF}, \tilde{\mu})$. Furthermore, using the definition of ϕ, and that $\underline{\mu}^1$ is implicitly defined through $\phi(\underline{\mu}^1) = 0$, one has $\underline{\mu}^1 = -\kappa_2 h(x(\mu^{LF}, \underline{\mu}^1))$. Therefore,

$$(1-\alpha)\bar{\theta} - \mu^{LF} - \underline{\mu}^1 = \tilde{\mu} - \underline{\mu}^1 = \kappa_2 \left(h(x(\mu^{LF}, \underline{\mu}^1)) - x(\mu^{LF}, \tilde{\mu}) \right). \tag{38}$$

Furthermore,

$$x(\mu^{LF}, \tilde{\mu}) = x(\mu^{LF}, \underline{\mu}^1) + \frac{\underline{\mu}^1 - \tilde{\mu}}{\sigma_2}. \tag{39}$$

Inserting 39 into 38, and rearranging, yields

$$\left(\tilde{\mu} - \underline{\mu}^1 \right) \left(1 + \frac{1}{2}\beta \right) = \kappa_2 \left(h(x(\mu^{LF}, \underline{\mu}^1)) - x(\mu^{LF}, \underline{\mu}^1) \right).$$

Recall that $h'(\eta) = h(\eta)[h(\eta) - \eta]$, that $h'(\eta) \in (0,1)$, and that $\lim_{\eta\to\infty} h(\eta) = \infty$. Hence, $h(\eta) > \eta$ and $\lim_{\eta\to\infty}(h(\eta) - \eta) = 0$. Since $\underline{\mu}^1 < 0$, $\lim_{\bar{\theta}\to-\infty} x(\mu^{LF}, \underline{\mu}^1) = \infty$, which implies that

$$\lim_{\bar{\theta}\to-\infty} \kappa_2 \left(h(x(\mu^{LF}, \underline{\mu}^1)) - x(\mu^{LF}, \underline{\mu}^1) \right) = 0.$$

As $1 + \frac{1}{2}\beta > 0$, this implies that $\lim_{\bar{\theta}\to-\infty}(\tilde{\mu} - \underline{\mu}^1) = 0$. Rewriting this last equality yields the lemma.■

In the body of the text we argued that $\Delta(\underline{\mu}, \underline{\mu}) < 0$ and that $\Delta(\mu^{LF}, \mu^{LF}) = 0$. Suppose that $\mu_i = \bar{\mu} > 0$. By definition, this means that if Player i gets bad news, she is indifferent between investing and not investing. Hence, one can think of her as someone who will never invest at time two—not even if the other player did so at time one. Hence, $\Delta(\bar{\mu}, \bar{\mu}) = \bar{\mu} > 0$. If equilibrium is unique, those results imply that $\Delta(\mu^c, \mu^c) > 0$ whenever $\mu^c > \mu^{LF}$. In turn, this implies that the first term of Equation 10 is negative if equilibrium is unique and if $\mu^c > (\mu^{LF}, \bar{\mu}]$. Recall from Proposition 1 that equilibrium is unique whenever $\bar{\theta} < 0$. It follows from Lemma 4 that if $\bar{\theta} \to -\infty$ the third term of 10 is non-positive. Hence, $\frac{dU}{d\mu^c} < 0$ if $\mu^c \in (\mu^{LF}, \bar{\mu}]$ and if $\bar{\theta}$ is sufficiently negative.

We are left to show that the social planner does not want to implement a $\mu^c > \bar{\mu}$ when the prior mean is sufficiently negative. From the explanations provided in the body of this paper, we know that if $\mu^c > \bar{\mu}$ all types between μ^c and $\underline{\mu}^0$ always invest at time two and that all types between $\underline{\mu}^1$ and $\underline{\mu}^0$ invest at time two only if they received good news. Hence,

$$\begin{aligned}
U &= \int_{\mu^c}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i + \delta \int_{\underline{\mu}^0}^{\mu^c} \mu_i f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i \\
&+ \delta \int_{\underline{\mu}^1}^{\underline{\mu}^0} \Pr(\mu_j > \mu^c | \mu_i) E(\theta | \mu_i, \mu_j > \mu^c) f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i. \\
&< \int_{\underline{\mu}^0}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i + \delta \int_{\underline{\mu}^1}^{\underline{\mu}^0} \Pr(\mu_j > \mu^c | \mu_i) E(\theta | \mu_i, \mu_j > \mu^c) f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i \\
&\equiv \overline{U}.
\end{aligned}$$

Observe that $\overline{U} = U$ when $\mu^c = \bar{\mu}$. Hence, it suffices to show that $\frac{d\overline{U}}{d\mu^c} < 0$ when $\mu^c > \bar{\mu}$ and when $\bar{\theta}$ is sufficiently negative. Taking into account the fact that $E(\theta | \underline{\mu}^0, \mu_j < \mu^c) = E(\theta | \underline{\mu}^1, \mu_j > \mu^c) = 0$, it follows that $\forall \mu^c > \bar{\mu}$,

$$\begin{aligned}
\frac{d\overline{U}}{d\mu^c} &= \frac{d\underline{\mu}^0}{d\mu^c}(1-\delta)\Pr(\mu_j > \mu^c | \underline{\mu}^0) E(\theta | \underline{\mu}^0, \mu_j > \mu^c) f\left(\frac{\underline{\mu}^0 - \bar{\theta}}{\sigma_\mu}\right) \\
&+ \delta \int_{\underline{\mu}^1}^{\underline{\mu}^0} \frac{\partial}{\partial \mu^c}\Big(\Pr(\mu_j > \mu^c | \mu_i) E(\theta | \mu_i, \mu_j > \mu^c)\Big) f\left(\frac{\mu_i - \bar{\theta}}{\sigma_\mu}\right) d\mu_i. \qquad (40)
\end{aligned}$$

It follows from Lemma 2 that $\underline{\mu}^0$ is implicitly defined by

$$\underline{\mu}^0 - \kappa_2 r\left(\frac{\mu^c - \alpha\underline{\mu}^0 - (1-\alpha)\bar{\theta}}{\sigma_o}\right) = 0.$$

It then follows from the implicit function theorem that

$$\frac{d\underline{\mu}^0}{d\mu^c} = -\frac{-\frac{\kappa_2}{\sigma_o} r'(\cdot)}{1 + \frac{\alpha\kappa_2}{\sigma_o} r'(\cdot)} = \frac{\frac{1}{1+\alpha} r'(\cdot)}{1 + \frac{1}{2}\beta r'(\cdot)} < 0,$$

where the inequality follows from the fact that $r'(\cdot) \in (-1, 0)$ and that $\beta \in [0, 1]$. Hence, the first term of Equation 40 is negative. It follows from Lemma 4 that the gain of waiting of the inframarginal types (with the exception of $\underline{\mu}^1$) is decreasing in μ^c when $\bar{\theta} \to -\infty$. Hence, The second term of 40 is non-positive. Hence, $\frac{d\overline{U}}{d\mu^c} < 0$ when $\mu^c > \bar{\mu}$ and when $\bar{\theta}$ is sufficiently negative.■

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