Incomplete Markets, Labor Supply and Capital Accumulation

Albert Marcet  Francesc Obiols-Homs  Philippe Weil†

Abstract

In this paper we explore the accumulation of capital in the presence of limited insurance against idiosyncratic shocks, borrowing constraints and endogenous labor supply. As in the exogenous labor supply case (e.g. Aiyagari 1994, Huggett 1997), we find that steady states are characterized with an interest rate smaller than the rate of time preference. However, we also find that when labor supply is endogenous the presence of uncertainty and a borrowing limit are not enough to give rise to “aggregate precautionary savings”.

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†A. Marcet, Universitat Pompeu Fabra and CEPR, F. Obiols-Homs, Centro de Investigación Económica, Instituto Tecnológico Autónomo de México, and P. Weil, Université Libre de Bruxelles, ECARE, CEPR and NBER.

Address for correspondence: Francesc Obiols-Homs, CIE-Instituto Tecnológico Autónomo de México. Av. Camino Sta. Teresa 930, México D.F.-10700, E-mail: obiols@itam.mx.
1 Introduction

A fundamental result in the precautionary savings literature is that capital market imperfections and the presence of uninsured idiosyncratic risk, leads agents to save more than they would if there were no uncertainty.\(^1\) This result holds in a wide variety of environments, including two periods and infinite horizon setups, partial and general equilibrium models, economies with production, and models with i.i.d. and correlated uncertainty. The typical comparison is between the outcome when agents receive random draws of labor income (or labor productivity endowments) from a given distribution, and the outcome when agents receive with certainty the expected income (or endowment) of that distribution. By construction, therefore, this notion of precautionary saving extends to the comparison between economies with and without complete markets (i.e., competitive insurance markets).\(^2\) In particular, Aiyagari (1994) and Huggett (1997) studied dynamic economies with production and showed that savings (hence capital accumulation) is larger under incomplete than under complete markets. We label this effect on capital accumulation the *Aiyagari-Huggett effect*. An immediate implication of the Aiyagari-Huggett effect is that output is larger under incomplete than under complete markets. This conclusion seems to give rise to an “incomplete markets puzzle”, because it is contrary to the informal intuition that a more efficient allocation of resources -possibly through a better developed financial intermediation- should not only increase welfare, but also output. In fact, there is an extensive theoretical and empirical literature studying the effects of financial institutions on the performance of industries and on growth rates. Just to cite an example, Levine (1997) uses data of a number of countries and reports that “there is a strong positive relationship between (...) financial indicators and (...) long run real per capita growth rates, capital accumulation and productivity growth” (Levine, 1997 pag. 706).\(^3\)

In this paper we explore the implications of a leisure/labor decision with respect to the Aiyagari-Huggett effect using a dynamic, general equilibrium model. In particular, we are interested in the effect of *limited insurance* against idiosyncratic risks

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\(^1\)See for instance Leland (1968) and Sandmo (1970), and more recently, Kimball (1990), Deaton (1991), Carroll (1991), Huggett (1993), Aiyagari (1994) and Huggett and Ospina (2001), among others.

\(^2\)In this paper we take precautionary savings as a larger accumulation of capital under uncertainty than under certainty rather than the notion associated to the convexity of marginal utility. See Huggett and Ospina (1997) for a recent study about the role of the third derivative of the utility function with respect to consumption.

\(^3\)The causality between financial development and economic growth seems to be still an unresolved question, as Levine (1997) himself emphasizes. Levine’s paper is a survey of theories and empirical results about these issues with many references, thus we address the interested reader to that paper. For a more recent treatment see Smith (2002).
on aggregate variables, such as capital and output, when labor supply is endogenous. Therefore, we compare equilibrium allocations between complete and incomplete markets economies. This clarification is important because equilibrium allocations under complete markets do not coincide with the allocations under no uncertainty once labor supply decisions are taken into account. In this paper, however, we will continue to use “precautionary savings” to denote a larger accumulation of capital under incomplete markets than under complete markets.

Endogenizing labor supply is interesting for several reasons. First, the time spent working is an important margin that households may use to adjust income fluctuations. For instance, there is substantial evidence of changes in the number of per capita hours worked in the U.S. over the post Korean War period. Second, abstracting from the labor/leisure decision precludes the analysis of wealth and substitution effects on labor supply. These effects are key in understanding aggregate consumption and saving (see for instance Benhabib, Rogerson and Wright (1991) and the related quantitative literature about real business cycles). Finally, to our knowledge very little is known about the effects on saving and output of limited insurance against idiosyncratic risks when labor supply is endogenous.

In section 2 we use a partial equilibrium model to analyze individual labor supply in the presence of idiosyncratic risk and limited insurance in a static environment. Under the assumption that leisure is a normal good, we show that incomplete insurance to idiosyncratic employment shocks introduces ex post wealth effects on labor supply. The implication of the ex post wealth effects on labor supply is that once uncertainty is realized, agents in the employment state under incomplete markets are richer than under complete markets (precisely because ex ante they could not buy an insurance against unemployment), and thus labor supply is smaller than under

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4 Barsky, Mankiw and Zeldes (1986, pg. 680), among others, report that “... individual incomes are highly vulnerable to disability, which includes medical, psychiatric, and other factors limiting hours of work or precluding work entirely”. If one takes the view that these types of shocks to labor productivity are unavoidable, it is natural to ask what are the differences, and what could be gained, with an improved, or complete, insurance scheme. We believe that this approach conforms well with the spirit of the precautionary savings literature.

5 See, for instance, Hansen (1985) and Kydland (1995). Although a major fraction of changes in hours of work is explained by changes in the employment rate -the extensive margin- changes in per capita hours worked -the intensive margin- explain about 20% of the total variation in hours. For studies using micro-data see Abowd and Card (1989) about the relationship between hours worked and earnings and the references therein.

6 At the theoretical level, Flodén (1998) studies the effect of endogenous labor in a two periods, partial equilibrium model in the tradition of the precautionary saving literature. Related quantitative results for dynamic, general equilibrium models can be found in Rios-Rull (1994) and Low (1998) for life cycle economies, and in Castañeda, Díaz-Giménez and Ríos-Rull (1996) and Obiols-Homs (2001), for models with infinitely lived agents. At the empirical level, the evidence regarding precautionary saving is far from clear. See for instance Carroll, Dynan and Krane (1999) and the references therein.
complete markets. This result suggests that introducing endogenous labor in a fully articulated model with imperfect insurances to idiosyncratic shocks can help to resolve the incomplete markets puzzle we mentioned above. Unlike static partial equilibrium models where the initial distribution of wealth and prices are exogenous, in a dynamic general equilibrium model the distribution of wealth is endogenous and has large effects on equilibrium prices. Therefore the main purpose in the following sections is to investigate how precautionary motives and labor supply fit together in a general equilibrium model.

In section 3 we develop a dynamic model with production and incomplete insurance markets to idiosyncratic shocks. The class of economies we analyze is similar to those in Imrohoroglu (1989), Aiyagari (1994), Huggett (1997) and Krusell and Smith (1998), among others. In these dynamic economies agents face idiosyncratic shocks to labor productivity and they inelastically supply their time endowment as labor. Another feature of these models is that agents are subject to a borrowing constraint. Although it is not essential for the results, we assume that there are only two idiosyncratic states, which we refer to as employment and unemployment. The main departure from the previous basic setup is that we assume that labor supply is elastic, and thus labor input has to be determined as part of the equilibrium.

We study first the decision problem of an infinitely lived agent that faces incomplete insurance to idiosyncratic shocks. We show that capital accumulation is bounded above even if the interest rate equals the rate of time preference. This is at odds with existing results in the incomplete markets literature abstracting from endogenous labor supply. The explanation for this result is that when leisure is a normal good, labor supply decreases with wealth. Thus, if the agent accumulates a sufficiently large amount of assets he will not supply any labor even if it is productive. This means that when labor supply is endogenous the uncertainty in labor income decreases in the level of wealth and eventually disappears. At that level of wealth, the agent has no incentive to increase asset holdings, even if the interest rate equals the rate of time preference. We then characterize competitive equilibria and steady states and study the effects of solvency and less than solvency borrowing limits. We show that steady states are such that the equilibrium interest rate is always smaller than the rate of time preference. Thus, we extend to endogenous labor the result previously obtained by Aiyagari (1994) and Huggett (1997) for production economies and by Huggett (1993) for exchange economies. Under standard assumptions on the technology for

\footnote{Jappelli and Pagano (1994) study the implications for growth of market imperfections in a deterministic model with overlapping generations. They find that a borrowing limit promotes higher growth rates. In relation to this, Hernández (1991) finds that a borrowing limit has no effects on steady states in his model without uncertainty and infinitely lived, heterogeneous agents. In the model we develop below, agents face both idiosyncratic uncertainty and a borrowing limit.}
production, an interest rate smaller than the rate of time preference implies that the
capital labor ratio is larger under incomplete markets than under complete markets.
Unlike with exogenous labor, where a larger capital labor ratio implies the existence
of precautionary saving and a larger output, the results are not so clear when labor
supply is endogenous. In particular, the existence of precautionary saving and a larger
output depends on the interaction between the Aiyagari-Huggett effect and the ex post
wealth effects on labor supply detected in the previous sections.

In section 4 we use numerical methods to investigate the interaction between ex post
wealth effects on labor supply and the Aiyagari-Huggett effect for several borrowing
limits. We provide an example of a steady state where there is no precautionary sav-
ings, even though there is a positive mass of constrained agents. Thus, ex post wealth
effects on labor supply seem to be able to reverse the Aiyagari-Huggett effect on cap-
tal accumulation and output. Our numerical results also suggest that the differences
between complete and incomplete markets tend to disappear as the borrowing limit
approaches the solvency limit.

The paper is concluded with some final comments in section 5 and an appendix with
proofs.

2 Wealth effects on labor supply in a static world

We begin our investigation of the relationship between market incompleteness and
labor supply in the simplest possible environment: a static economy in which there is
a probability that a fraction of the population might be unemployed for technological
reasons. Our objective is to answer a basic, but fundamental question: does the
unavailability of unemployment insurance lead people to work on average more or
less than when unemployment insurance is available? In other terms, does market
incompleteness increase or reduce work effort? The answer, as we shall see, relies
heavily on wealth effects.

2.1 A simple model

For these questions to make sense, we must of course posit, to start with, that labor
supply is elastic with respect to the wage (work effort would otherwise by definition
be the same, for employed workers, under complete and incomplete markets). Let’s
assume that the economy consists of a continuum of consumers over the unit interval,
and that all consumers have identical preferences $U(c, l)$ over consumption and leisure.
The utility function $U(\cdot, \cdot)$ is strictly increasing and strictly concave in each of its
arguments, and we assume that it satisfies the Inada conditions
\[ \forall l \geq 0 : \lim_{c \to 0} U(c, l) = +\infty, \quad \forall c \geq 0 : \lim_{l \to 0} U(c, l) = +\infty. \]

In addition, we require that \textit{leisure be a normal good.}

All consumers are endowed with one unit of leisure, but are subject to exogenous idiosyncratic labor productivity shocks: when a consumer wakes up in the morning, she is either sick or healthy.\(^8\) We assume that sickness (which we call the bad state of nature) makes work impossible, and we therefore model labor productivity \(s\) as a random variable
\[ s = \begin{cases} 0, & \text{with probability } 1 - \phi; \\ 1, & \text{with probability } \phi. \end{cases} \]

We shall usually refer to sick workers as \textit{unemployed} workers, although a more accurate, but more awkward, description would be “unemployable”. The employment probability \(\phi\) is the same for all agents, but the ex post realization of the employment process is individual-specific. In the aggregate, given the continuum assumption, \(\phi\) also measures the fraction of the population that is employed.

Consumers receive a non-produced endowment \(\Omega\) of the consumption good. This endowment is non-random, and is identical across agents. Consumers can supplement their endowment of the consumption good by devoting some of their leisure to work: one (efficiency) unit of work produces 1 unit of the consumption good. Obviously, only healthy consumers, who are productive, will ever choose to sacrifice leisure to work. A sick consumer who would attempt to work would provide zero efficiency units of labor, and therefore get a zero labor income in return for his valuable leisure - hardly an optimum tradeoff. In short, using our slightly abusive terminology, only the employed agents work.

Finally, we assume that there is no aggregate risk.

### 2.2 Complete markets

Suppose that the consumers’ state of health can be perfectly and costlessly monitored by third parties (imagine, for instance, that chickenpox is the disease that afflicts consumers and makes them unable to work). In that setup, competitive insurance companies can offer unemployment insurance to the consumers at the actuarially fair price
\[ p = 1 - \phi. \tag{1} \]

By paying \(p\) units of the consumption good to the unemployment (or health) insurance company, consumers buy the right to get 1 unit of the consumption good if they end

\(^8\)consumer’s actions. We thus abstract for moral hazard considerations.
up unemployed at the end of the period. But how much insurance will consumers buy? How does the demand for insurance interact with consumption and labor supply decisions?

Let $Q$ denote the demand for insurance. A consumer with non-produced endowment $\Omega$ solves the following program:

$$\max_{c^e, c^u, Q} \phi U(c^e, l) + (1 - \phi)U(c^u, 1)$$

subject to

$$c^e + pQ = \Omega + (1 - l), \quad \text{(2)}$$
$$c^u + pQ = \Omega + Q, \quad \text{(3)}$$
$$c^e, c^u, Q \geq 0, \quad 0 \leq l \leq 1. \quad \text{(4)}$$

Here $c^e$ and $c^u$ denote consumption in the employment and unemployment states, and $l$ denotes leisure in the employment state. Leisure in the unemployment state is, by construction, equal to its maximum possible value 1: the unemployed have no labor income, but they do rest.

Using the fact that $p = 1 - \phi$, the first-order conditions for an interior\(^9\) solution can be written compactly as

$$U_c(c^u, l) = U_c(c^e, 1), \quad \text{(5)}$$
$$U_l(c^e, l) = U_c(c^e, l). \quad \text{(6)}$$

Equation 5 describes, given labor supply, the optimal insurance decision: equalize the marginal utility of consumption in the unemployment and employment states (note that this does not require setting $c^u = c^e$; this would be the case with $U(\cdot, \cdot)$ separable in $c$ and $l$, and the results below would still go through). Equation 6 characterizes, given the amount of insurance bought by the consumer, her optimal consumption and leisure decisions: equalize the wage rate (here set to 1) to the marginal rate of substitution between leisure and consumption $U_l/U_c$.\(^{10}\) Together with the budget

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\(^9\)The Inada conditions guarantee that $c$ and $l$ are strictly positive at the optimum. To make sure that the constraint $l \leq 1$ is not binding, we need only impose (since leisure is assumed to be normal) that $\Omega$ be not too large. When $\Omega$ is very large, the constraint $l \leq 1$ binds and uncertainty has no effects: our consumer chooses never to work, regardless of whether she is employed or not. This case will play an important role in the next section when we deal with an infinite horizon model. In such a model, the consumer's wealth becomes endogenous and we have to pay attention to the possibility that a consumer may become so wealthy as to stop working. In the present static setting, however, we disregard corner solutions to better focus on the role of wealth effects.

\(^{10}\)One can readily verify that the implied consumption allocation coincides with the command optimum $\max_{c^e, c^u, l} U(c^e, c^u, l)$ subject to $\phi c^e + (1 - \phi)c^u = l$. The competitive equilibrium we describe is indeed a complete market equilibrium.
constraints (2) and (3), these two-first order conditions enable us to compute the optimal \((c^e, c^u, l, Q)\) under complete markets.

We now establish a basic, and intuitive, property of the consumer’s insurance decision: 

**Lemma 1:** If leisure is a normal good, then the demand for insurance is strictly positive \((Q > 0)\) in an interior solution with \(l < 1\).

**Proof:** See the Appendix.

### 2.3 Incomplete markets

Our consumer’s problem is much simpler when she cannot insure against individual productivity shocks. When she is unemployed, she just consumes her endowment:

\[ c^{iu} = \Omega, \]

where variables superscripted \(i\) hereafter denote incomplete market magnitudes. When she is employed, our consumer chooses consumption and work effort \((c^{ie}, l^i)\) that solve the following problem:

\[
\begin{align*}
\max_{c^{ie}, l^i} & \quad U(c^{ie}, l^i) \\
\text{subject to} & \quad c^{ie} = \Omega + (1 - l^i), \quad (7) \\
& \quad c^{ie} \geq 0, \quad 0 \leq l^i \leq 1. \quad (8)
\end{align*}
\]

The first-order condition for an interior solution\(^{11}\) is

\[ U_l(c^{ie}, l^i) = U_c(c^{ie}, l^i). \quad (9) \]

Together with the budget constraint (7), this first order condition enable us to compute the optimal \((c^{ie}, l^i)\) under incomplete markets.

### 2.4 Labor supply under complete and incomplete markets

We are now ready to answer the question we asked at the beginning of this section: does market incompleteness lead consumers to work more or less than they would under complete markets? The answer is provided by

\(^{11}\)We abstract, as we did in the complete market case, from circumstances in which \(\Omega\) is so large that the consumer wishes not to work even when she is employable.
Proposition 1: If leisure is a normal good (and for interior solutions), labor supply is lower under incomplete markets than under complete markets (i.e., \( l < l^i < 1 \)).

Proof: See the Appendix.

This proposition says something obvious: in the same way that you are better off ex post if your house does not burn and you have not bought fire insurance, consumers who do end up working are richer ex-post under incomplete markets than under complete markets—because they did not pay an insurance premium for an unrealized contingency!\(^{12}\) Therefore, when labor supply is elastic, there is a fundamental economic mechanism—wealth effects in labor supply—that tends to “shrink” the size of incomplete market economies relative to complete markets. When leisure is a normal good, the employed are richer ex-post, and thus work less, when markets are incomplete!

Models that compare the “size” of incomplete market economies to the no uncertainty analogue while assuming that labor supply is fixed, such as Aiyagari (1994) and Huggett (1997), conclude that incomplete market economies are “larger” than the no uncertainty counterparts (the level of the capital stock and of output are higher in the former). These models by construction abstract from the very wealth effects on which we have focused so far. Wealth effects in labor supply exert, when labor supply is elastic, a “shrinking” force on the size of the economy. This force runs counter to the “enlarging” mechanisms (precautionary and/or buffer stock saving) at the heart of the Aiyagari-Huggett effect. It is therefore natural to ask—and this is the main interrogation of the following sections—how wealth effects in labor supply may interact, in a multi-period model with endogenous wealth distribution, with intertemporal savings decisions. What is the end result of these conflicting forces on the equilibrium capital-labor ratio and on equilibrium output? Are incomplete market economies “larger” or “smaller” than their complete market counterparts?

3 Dynamic model with endogenous effort

In this section we describe a benchmark model with infinite horizon and incomplete insurance to idiosyncratic shocks.\(^{13}\) Since the model is close to the one in Huggett (1997), we keep his notation and draw on some of his results.

\(^{12}\)It is worth emphasizing that this statement is most definitely only a statement about ex post, and not ex-ante, welfare. But it is ex post wealth that matters for labor supply decisions.

\(^{13}\)In our approach there is a large number of states of nature and only one asset. Although in equilibrium asymmetric information is known to prevent the existence of some assets (see Green (1987), for instance) we follow the related literature and limited insurance is introduced as an assumption.
3.1 Preferences, endowments and technology

The economy is inhabited by a continuum of agents in the unit interval. These agents behave so as to maximize the expected value of discounted utility, which depends on infinite sequences of consumption and leisure, \( \{c_t, l_t\}_{t=0}^{\infty} \). We assume that all agents have the same preferences and discount factor, thus they maximize:

\[
E \left[ \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \right], \text{ where } 0 < \beta < 1.
\]  

(10)

The per period utility is restricted to satisfy the following assumptions:

- **A1**: \( U: A \subset \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \), is continuous and differentiable.
- **A2**: \( U(\cdot, \cdot) \) is strictly increasing and strictly concave in each of its arguments, with \( U(0, l) = U(c, 0) = 0 \), and \( \forall l \geq 0 : \lim_{c \rightarrow 0} U(c, l) = +\infty \), \( \forall c \geq 0 : \lim_{l \rightarrow 0} U(c, l) = +\infty \).
- **A3**: \( U(\cdot, \cdot) = u(c) + n(l) \) and is homogeneous of degree \( 0 < \gamma < 1 \).

Assumptions 1 and 2 are standard in the literature. Assumption 3 is not essential for the results and is adopted to simplify as much as possible the analysis of the allocation of time between leisure and labor.

At the beginning of every period, each agent is endowed with a unit of time and a labor productivity endowment \( s \in S \). Labor productivity endowments follow a Markov chain with probabilities of transition \( \pi_{s' | s} \). Since there is a continuum of agents, this means that there is uncertainty at the individual level but that there is no aggregate uncertainty. Under the additional restrictions in A4, agents in state \( s = 1 \) will be fully productive and they will be able to choose the number of hours they supply as labor to the market in exchange of the wage rate \( w \). Following the convention in the previous section, agents in state \( s = 0 \) can be thought of being sick, out of the labor market, or simply unemployed.

- **A4**: \( S = \{1, 0\} \) with \( \sum_{s'} \pi_{s' | s} = 1 \) and \( \pi_{s' | s} > 0 \) for all \( s, s' \in S \). Furthermore, \( \pi_{1|1} \geq \pi_{1|0} \).

Under the interpretation of labor productivity endowments as employment shocks, we can denote by \( \phi_t \) the employment rate at any period \( t \). Given A4, the employment
rate satisfies the recursion \( \phi_t = \phi_{t-1} \pi_{1|1} + (1 - \phi_{t-1}) \pi_{1|0} \), and since we will consider only steady states, \( \phi_t \) is given by:

\[
\phi_t = \frac{\pi_{1|0}}{1 - \pi_{1|1} + \pi_{1|0}}, \forall t \geq 0.
\]

The production side of the economy is standard. Output in period \( t \) is given by an aggregate production function \( f(K, H) \). This function relates previously accumulated units of capital and current labor and is restricted to satisfy the assumptions in A5:

- A5: \( f(K_{t-1}, H_t) = F(K_{t-1}, H_t) + (1 - \delta)K_t \), where \( \delta \in (0, 1) \) is the depreciation rate and \( F \) is homogeneous of degree one, with \( F(K, 0) = F(0, H) = 0 \), \( F_j > 0 \) and \( F_{j,j} < 0 \), for \( j = K_t, H_t \).

3.2 The decision problem of an agent

Each agent in the economy solves a version of the “income fluctuations problem” extended to include a labor/leisure decision.\(^\text{14}\) More precisely, in the current setting an agent faces a deterministic sequence \( \{w, r\}_{t=0}^\infty \) of wage rates and interest rates and chooses her labor supply and asset holdings over time so as to maximize the discounted expected utility in Eq. (10). We assume that there is only one asset (capital) which may be used as a buffer to smooth out consumption in the unemployment state. Capital holdings are subject to a borrowing limit \( B \) which prevents agents to have access to complete insurance by issuing an arbitrarily large amount of debt. Formally, assuming that \((w, 1+r)\) are strictly positive and constant through time, an agent solves the following optimization problem, where the expectation is over the idiosyncratic shocks:

\[
\max_{c_t, l_t, k_t} E \left[ \sum_{t=0}^\infty \beta^t U(c_t, l_t) \right],
\]

s. to
\[
\begin{align*}
0 \leq c_t, & \quad c_t + k_t \leq s_t w(1 - l_t) + (1 + r)k_{t-1}, \\
0 \leq l_t \leq 1, & \quad 0 \leq B \leq k_t \leq \bar{B}, \quad B \leq k_{-1} \text{ given.}
\end{align*}
\]

We will study the previous problem using dynamic programming techniques. Since the utility function is unbounded above, to prepare the way we need to introduce an upper bound \( \bar{B} \) on capital holdings. With the state space so restricted the utility function is bounded and this enables us to use well known results of dynamic programming.\(^\text{15}\)

\(^{14}\)See for instance Schechtman and Escudero (1977) and Clarida (1987).

\(^{15}\)We show in the appendix that capital holdings are bounded above for the relevant range of interest rates, thus with an appropriate choice of \( B \) there is no loss of generality.
Notice that the borrowing limit $\bar{B}$ is assumed to be non negative. Since in the unemployment state an agent receives no labor income, $\bar{B} = 0$ would in fact set the borrowing limit equal to the solvency level.\footnote{The solvency level is given by the amount $k\bar{B}$ of capital that once the stock of capital is at that level, if the agent receives the worst productivity endowment all resources cover exactly up to keep that level of capital and, therefore, consumption is zero (see for instance Huggett and Ospina (1997)). Aiyagari (1994) shows that this sort of borrowing limit is implied by present value budget balance.} Notice also that the previous problem may not have a solution if the borrowing limit is strictly positive and the interest rate is negative. In the analysis below, where we study the effects of solvency and less than solvency borrowing limits, it will be understood that the budget set is non empty.

To state the problem of the agents in the language of dynamic programming we need to introduce some additional notation. The position at a point in time of an agent is described by $x = (k, s)$, i.e., her asset holdings and productivity endowment. For given prices $(w, r)$, individual states are restricted to belong to the state space $X = [\underline{B}, \bar{B}] \times S$. To further simplify the problem we use the facts that, (i) $A2$ implies that leisure is strictly positive in all states, and (ii) $A3$ implies a linear relationship between consumption and leisure in the employment state.\footnote{To see this, notice that $A3$ implies that $n(l) = l''n(1)$ and $u(c) = c''u(1)$. Let $l^*$ and $c^*$ stand for the optimal controls of the utility maximization problem of an employed agent and assume an interior solution for leisure, thus they satisfy $u'(c^*)w = n'(l^*)$. Using the previous expressions we obtain $l^* = b(w)c^*$, where $b(w) = (n(1)/wu(1))^{1/(1-\gamma)}$, which is well defined as long as $w, u(1)$ and $n(1)$ are strictly different from zero. We show in the appendix that $\bar{B}$ can be chosen to be the smallest amount of capital that would lead an employed agent to supply no labor at the given wage and interest rate.}

With the previous notation and substituting the linear relationship between consumption and leisure in the objective and budget constraint, an agent solves the following dynamic programming problem, where $v$ is the value function:

$$v(x; w, r) = \sup_{k' \in \Gamma(x; w, r)} \{U(x, k') + \beta E[v(x'; w, r)|x]\},$$

where

$$U(x, k') = \begin{cases} u((1 + r)k - k'), & \text{if } s = 0, \\ u\left(\frac{w + (1+r)k - k'}{1 + wb(w)}\right) + n\left(\frac{b(w)(w + (1+r)k - k')}{1 + wb(w)}\right), & \text{if } s = 1, \end{cases}$$

and

$$\Gamma(x; w, r) = \{k' : B \leq k' \leq \min\{sw + (1+r)k, \bar{B}\}\}.$$
non standard result is $R_4$ (the notation is simplified by eliminating prices as arguments in the value function and decision rules).

Remarks: Assume A1-A4, $(w, 1 + r) > 0$ and $\beta(1 + r) \leq 1$ for all $t$, then:

$R1$: $v(x)$ is strictly increasing and strictly concave in $k$, and $c(x), l(x)$ and $k(x)$ are continuous in $k$.

$R2$: $c(x), l(x)$ and $k(x)$ are strictly positive, $c(k, s)$ and $l(k, 1)$ are strictly increasing in $k$ and $k(k, s)$ is increasing in $k$.

$R3$: For all $k \in [\underline{B}, \overline{B}], k(k, 0) \leq k$ (with strict inequality if $\underline{B} < k < \overline{B}$).

$R4$: $\overline{B}$ can be chosen so that for some $\bar{k} < \overline{B}, k(\bar{k}, 1) \leq \bar{k}$.

Proof: See the Appendix.

The result in $R_4$ together with the continuity of the decision rules in $R1$ and monotonicity in $R2$, a sufficient condition that guarantees $l(k, 1) \leq 1$ is that $k_{-1}$ lies in $(\underline{B}, \overline{B})$. In the remainder of the paper we will assume that this is so. Notice also that $R_4$ implies that even if $\beta(1 + r) = 1$, individual capital accumulation remains bounded above. Figure 1 displays the typical shape of decisions rules for capital accumulation assuming the interest rate is equal to the rate of time preference. The figure reflects the full extent of the wealth effect we mentioned in the introduction: since leisure is a normal good ($R2$), if an agent becomes sufficiently rich she will not supply any amount of labor even if by doing so she could increase her consumption in the future. Therefore, at that level of capital, the agent faces no uncertainty, hence for any interest rate smaller or equal to the time preference rate, the opportunity cost of increasing capital accumulation exceeds the discounted benefit of increased consumption in the future. This is a fundamental difference with respect to other incomplete markets models where labor is exogenous. In those models if $\beta(1 + r) = 1$ capital accumulation is unbounded, and the only way to prevent an infinite accumulation is to fix an interest rate smaller than the rate of time preference.

3.3 Firms

We assume there is a single profit maximizing firm operating the technology for production. As usual, we think of this firm as representing many identical small firms operating in competitive markets, thus the following equations will hold in equilibrium:

$$f_{K_{t-1}} = (1 + r_t), \quad f_{H_t} = w_t, \forall t \geq 0.$$ (16)
3.4 Equilibrium

Since the lack of complete insurance markets will give rise to heterogeneity among consumers, the definition of the recursive competitive equilibrium necessarily includes a description of that heterogeneity and its evolution over time.\footnote{The reason for this is that, in general, equilibrium prices depend on the distribution of wealth, since agents with different wealth will supply different amounts of labor and will have different propensities to save. See, for instance, the discussion in Ríos-Rull (1994).}

Let $\psi_t$ be the aggregate state describing at any period $t$ the mass of agents in each possible state. Thus the aggregate state is a probability measure defined on the sets $\mathcal{X}$, the Borel subsets of $X$. We will denote $P(x, t, A)$ the transition function giving the probability that a worker in individual state $x$ at time $t$ will have an individual state that lies in the set $A \in \mathcal{X}$ next period.\footnote{For a construction of the transition function, see Theorem 9.13 in Stokey and Lucas (1989, pg. 284). See also Huggett (1993) and Krusell and Smith (1998) for different numerical implementations.}

The two functions and its relationship are provided below:

\[
P(x, t, A) = \pi(\{s' \in S : (k(x, t), s') \in A\}|s),
\]

\[
\psi_{t+1}(A) = \int_X P(x, t, A) d\psi_t \text{ for all } A \text{ in } \mathcal{X}.
\]

To complete the characterization of the environment we need to define two additional functions, $K(\psi)$ and $H(\psi)$, indicating the aggregate stock of capital and labor as a function of the aggregate state $\psi$. We are now ready to define the recursive competitive equilibrium of incomplete markets corresponding to the economy presented above.

**Definition:** A stationary recursive competitive equilibrium with incomplete markets for the economy described above is a list of functions $(v, c, l, k)$ and a list $(w, r, K, H, \psi)$ such that:

1. $v$ satisfies the functional equation in Eq. (13) and $c(x)$, $l(x)$ and $k(x)$ are the associated optimal decision rules given $(w, r)$.
2. $(w, r)$ satisfy Eq. (16) for all $t \geq 0$.
3. Market clearing: for all $t \geq 0$,
   \[
   \int_X (c(x) + k(x)) d\psi = f(K(\psi), H(\psi)),
   \]
4. Aggregate factor inputs are generated by decision rules of the agents:
   (i) $\int_X k(x) d\psi = K(\psi)$.
   (ii) $\int_X s(1 - l(x)) d\psi = H(\psi)$.
5. Law of motion: $\psi(A) = \int_X P(x, A) d\psi$ for all $A$ in $\mathcal{X}$. 
The following result extends Theorem 1 in Huggett (1997) and establishes the relationship between the rate of time preference and the interest rate in a stationary equilibrium.

Proposition 2: Under A1-A5 and \( B \geq 0 \), in a steady state with positive capital and labor \( \beta(1 + r) < 1 \).

Proof: See the Appendix.

The case of \( B > 0 \) corresponds to a borrowing limit smaller than solvency and is the standard case in the literature about precautionary saving. If there exists a stationary equilibrium with such a borrowing limit then there is a positive mass of agents who are borrowing constrained, thus the Euler Equation for these agents holds with inequality. Since at a steady state any statistic of aggregate states must remain constant over time, average marginal utility of consumption will be constant only if the interest rate is strictly smaller than the rate of time preference. The case of \( B = 0 \) corresponds to the borrowing limit implied by a solvency constraint. In this case the Euler Equation of the consumer’s problem holds always with equality, and the previous argument cannot be directly applied. As in the exogenous labor income case, the presence of uncertainty leads agents in the good state to increase asset holdings even if the interest rate equals the rate of time preference. However, the richer the agent the smaller his labor supply, thus labor income uncertainty decreases with wealth. In particular, if the interest rate equals the rate of time preference, then in the long run all agents are sufficiently rich so that non of them provides any time as labor. This means that steady states characterized with an interest rate equal to the rate of time preference are not possible. Thus if a steady state exists, the interest rate has to be smaller than the rate of time preference. At any of those steady states the distribution of wealth is uniquely determined.

The previous proposition does not say anything about whether capital and labor, hence output, will be larger or smaller under incomplete markets than under complete markets. Before we proceed with this comparison we need to characterize the the steady state under complete markets. Using the fact that the competitive equilibrium with complete markets corresponding to the previous economy is efficient, the simplest way to determine the competitive allocation is to solve the planner’s problem. With the optimal allocation, equilibrium prices of inputs are determined using the first order conditions of the firm. It is straight forward to show that a steady state with complete markets is such that \( r = (1 - \beta)/\beta \). Therefore the previous proposition implies that a steady state the capital labor ratio under incomplete markets is larger than under complete markets. This fact, however, does not necessarily imply that

---

20The arguments in Huggett (1993), based on Theorem 2 in Hopenhayn and Prescott (1992) can be readily applied to the endogenous labor case considered in this paper.
there is “precautionary saving”. Unfortunately, we have been unable to determine under what conditions the wealth effects on labor supply will dominate the Aiyagari-Huggett effect, thus in the following section we investigate this issue using numerical methods.

4 Ex post wealth effects vs. the Aiyagari-Huggett effect

In this section we compare aggregate variables of the incomplete markets economy to those predicted by standard representative agent with complete markets theory. The reason for this choice is that unlike the stationary equilibrium with incomplete markets, at a steady state with complete markets the distribution of wealth is not uniquely determined. Thus there are redistributions of wealth that would imply sizable differences in output and aggregate capital without changing equilibrium prices. It would not be appealing to base the comparison of equilibrium allocations across market arrangements on particular distributions for which we have no theory.

Unlike the exogenous labor case, it is worth mentioning that when labor supply is endogenous the solution with complete markets does not coincide with the solution where all agents receive with certainty the expected value of the labor productivity endowment, i.e., without uncertainty. This follows from the strict concavity in the utility from leisure.

The model we use in the simulations is a version of the divisible labor model in Hansen (1985) without aggregate uncertainty. Preferences are given by: $u(c, l) = \log(c) + A\log(l)$; and the production function is Cobb-Douglas: $f(K, H) = TK^\alpha H^{1-\alpha} + (1-\delta)K$. We keep the assumption of only two states for the idiosyncratic shocks, and the parameter values describing preferences and technologies are presented in Table 1. Except for $T$ (which is smaller than 1 to reduce the state space) this calibration is consistent with standard exercises in the real business cycle literature simulating quarterly data.

Table 1: Parameter values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.99</td>
</tr>
<tr>
<td>$A$</td>
<td>2</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.36</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
</tr>
<tr>
<td>$T$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

To solve for an equilibrium we proceed in three iterative steps. In the first step

---

21Chatterjee (1994) and Caselli and Ventura (2000) provide a theory of transitional dynamics of the income/wealth distribution of a deterministic model without the labor/leisure decision. In these studies the equilibrium distribution at the steady state is essentially determined by initial conditions.
Table 2: Equilibrium allocations.

<table>
<thead>
<tr>
<th></th>
<th>K</th>
<th>H</th>
<th>K/H</th>
<th>Y</th>
<th>sr</th>
<th>r - δ</th>
<th>k_{max}</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.M.</td>
<td>2.9746</td>
<td>.2312</td>
<td>12.8617</td>
<td>3.1903</td>
<td>.2564</td>
<td>.0101</td>
<td></td>
</tr>
<tr>
<td>I.M.</td>
<td>2.9689</td>
<td>.2287</td>
<td>12.9801</td>
<td>3.1825</td>
<td>.2579</td>
<td>.0098</td>
<td>15.4747</td>
</tr>
</tbody>
</table>

we fix a capital labor ratio and find the interest rate and wage rate as indicated in Eq. (16). Given these prices we approximate decision rules for capital accumulation, consumption and leisure on a grid of 800 points. Between grid points we use linear interpolation, and we use a finer grid near the borrowing limit where decision rules are likely to be non linear. In the second step we compute aggregate capital and labor implied by the optimal decision rules and determine the corresponding capital labor ratio. To perform these calculations we determine the invariant distribution of agents over states implied by the decision rules for capital found in the previous step. 

This capital labor ratio is the input in a bisection (as explained for instance in Aiyagari (1994)) that in the third step updates the interest rate and wage rate to be used in the next iteration. We repeat steps one to three until the market clearing for capital is approximately zero. The equilibrium with complete markets can be solved directly using the first order and equilibrium conditions.

Table 2 reports aggregate capital, hours worked, the capital labor ratio, output, saving rate ($sr$) and net real interest rate corresponding to the equilibrium allocation under complete and incomplete markets corresponding to a case where $\pi_{11} = \pi_{10} = 0.5$, and $B = 1.25$. $k_{max}$ is the intersection of the decision rule for capital in the employment state with the 45 degrees line. For the reported allocation market clearing for capital is within .0021 units of zero and interest rates vary less than $10^{-5}$. We use a grid where the distance between points is between .0015 and .035. It is clear from the table that aggregate capital, labor and output are smaller under incomplete markets and that the capital labor ratio and the savings rate are larger than under complete markets.

In this example wealth effects on labor supply dominate the Aiyagari-Huggett effect. Therefore, this is an example where there is uninsurable idiosyncratic risk, a positive borrowing limit and yet, there is no “precautionary saving”. In the previous economy, completing the markets would not only increase welfare, but also output would rise. 

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22 These are the methods used in Huggett (1993) and are explained in detail in that paper. The first part of the process amounts to iterate on the derivatives of the value function and approximates optimal decision rules for capital accumulation by piece wise linear functions which hold exactly at points in the grid. The decision rule for capital is determined simultaneously with that of consumption and labor supply. In the second part of the process we have computed also the integration by simulation exploiting ergodicity and we obtain very similar results.

23 In regards to fiscal policy this example suggests that taxing capital income may be a “bad idea”
Figures 2 and 3 display the decision rules for capital and the equilibrium distribution. In figure 2 the decision rules for capital appear to be remarkably close to the 45 degree line. Except near the borrowing limit there is no strong incentive to sacrifice current consumption for a larger accumulation of capital when the agent is employed, and when the agent is unemployed the stock of capital decreases very slowly. Figure 3 reveals that even though the upper bound of the ergodic set for capital is above 15 units, most agents hold less than one third of that amount.\footnote{The kink in the decision rules for capital in Figure 2 is due to the increase of the distance between the points in the grid. In Figure 3 there appears only one line because \( \pi_{1|1} = \pi_{1|0} = 0.5 \) thus for all levels of capital there is the same mass of agents with \( s = 0 \) and \( s = 1 \). Also, the distribution appears to be flat for capital stocks larger than 5 units. This is due to inaccuracies of the figure's editor. In fact, we double checked the results and computed the aggregate variables by simulation over 300,000 periods and obtained the same results.}

It is relatively simple to compute other examples where the ex-post wealth effects on labor supply dominate the Aiyagari-Huggett effect. We have also experimented with a version of the previous model using a standard calibration for the U.S. economy. In this case we fix \( T = 1, \pi_{e|e} = 0.94 \) and \( \pi_{e|u} = 0.91 \). The calibration for the probabilities of transition is similar to the one in Imrohoroglu (1989) and approximately matches the 93\% average employment rate (after normalizing with the participation rate) and the 13 weeks of average duration of unemployment observed in the U.S. economy since the Korean War. Table 3 reports a few examples corresponding to the previous calibration under different borrowing limits. In the allocations reported in Table 3 equilibrium interest rates vary less than \( 10^{-6} \), and market clearing is within .001 units of zero. The results in the table suggest that as we decrease the borrowing limit, the differences between complete and incomplete markets become also smaller.

The computation of the equilibrium of the previous economy becomes increasingly difficult as we reduce the borrowing limit. The main problem we find is that in the numerical algorithm decision rules are not very sensitive to changes in the interest rate when it is close to the rate of time preference. For a capital labor ratio implying

\begin{table}[h]
\centering
\caption{Equilibrium allocations.}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
          & \( K \) & \( H \) & \( K/H \) & \( Y \) & \( sr \) & \( r - \delta \) & \( k_{\text{max}} \) \\
\hline
I.M. \( B = 8 \) & 11.2659 & .2951 & 38.1748 & 12.0793 & .2572 & .0099 & 24.1175 \\
I.M. \( B = 7 \) & 11.2298 & .295 & 38.063 & 12.0426 & .2567 & .01005 & 34.1187 \\
\hline
\end{tabular}
\end{table}
an interest rate slightly below \((1 - \beta)/\beta\) (and a wage rate slightly above the one under complete markets) we usually find that under incomplete markets the aggregate amount of capital is smaller and the aggregate labor supply is larger than their counterparts under complete markets. This results in a capital labor ratio too small to be sustained as a stationary equilibrium of incomplete markets. Notice also that in Table 3 the state space for capital increases as we decrease the borrowing limit, i.e., the \(\bar{k}\) such that \(k(k, 1) = \bar{k}\) is an increasing function of \(B\). To keep computational costs within reasonable bounds we use a coarser grid. The problem with this is that we lose accuracy in the decision rules, and it is not possible to compute an accurate approximation of the competitive equilibrium even for moderately small borrowing limits. To conclude this section with a more precise picture of these facts, in the above economy with a borrowing limit equal to 5.5 the decision rules corresponding to a gross interest rate equal to .03510099, imply \(K = 11.0256\) and \(H = .29549\), which produce a capital labor ratio \(k = 31.3128\) (more than 6 units below the ratio under complete markets) and \(r = .0355\), well above the gross interest rate \(r = .035101\) of equilibrium under complete markets.

5 Concluding remarks

In this paper we explore the implications of endogenous labor supply in a general equilibrium model with incomplete markets against idiosyncratic shocks and borrowing limits. We show that steady states are characterized by an interest rate smaller than the rate of time preference. This result, as in the exogenous labor supply case studied previously in the literature, implies that the capital labor ratio under incomplete markets is larger than under complete markets. Using numerical methods we show that steady states with a stock of capital and output smaller than under complete markets are possible with endogenous labor. Thus incomplete markets against idiosyncratic shocks and the presence of borrowing limits do not necessarily imply precautionary savings since labor supply decisions are taken into account. We also show than the differences between the two market arrangements tend to be smaller as the borrowing

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25 The problem of insensitivity of decision rules also appears in the other methods we have tried. In fact, we started solving the previous models with the Parameterized Expectations Approach developed by Den Haan and Marcet (1992). With PEA we found that the decision rules for capital of employed and unemployed agents crossed above the 45 degree line, which cannot happen according to theory. We managed to solve this inconvenient by introducing lower bounds for consumption of unemployed agents, but then we detected that the integration by simulation was rather unstable as we increased the number of periods in the series (even for long simulations). For this reason we first switched to value function iterations. This method was discarded because the problem of insensitivity of decision rules was very severe.

26 For the versions of the model in Table 3 we use an exponential grid of 950 points.
limit approaches the solvency limit and the mass of constrained agents decreases.
References


CHATTERJEE, S., 1994. *Transitional Dynamics and the Distribution of Wealth in a*


6 Appendix

Proof of Lemma 1: Consider the problem \( \max_{c,l} U(c,l) \) subject to \( c = \omega + (1-l) \). The first-order condition for an interior solution is \( U_l(c,l) = U_c(c,l) \). Manipulation of this first-order condition establishes that leisure is a normal good if and only if

\[
\frac{\partial l}{\partial \omega} = \frac{U_{cc} - U_{cl}}{U_{cc} - 2U_{cl} + U_{ll}} > 0.
\]

Since the second-order condition for a maximum imposes that the denominator of this expression be negative, normality of leisure requires that

\[
U_{cc} - U_{cl} < 0. \tag{17}
\]

But then, combining (1) with the budget constraints (2) and (3), and inserting them into the first-order condition (5) that governs the optimal insurance decision, we have:

\[
U_c[\Omega + \phi Q, 1] = U_c[\Omega - (1 - \phi)Q + (1 - l), l] < U_c[\Omega - (1 - \phi)Q, 1], \tag{18}
\]

where the second inequality follows from (17) and \( l < 1 \). Now \( U_{cc} \) is strictly negative by assumption. Therefore (18) implies that \( Q > 0 \).

Proof of Proposition 1: From equations (1), (2) and (6), labor supply under complete markets \( l \) satisfies

\[
U_l[\Omega - (1 - \phi)Q + (1 - l), l] = U_c[\Omega - (1 - \phi)Q + (1 - l), l], \tag{19}
\]

while, using (7) and (9), labor supply under incomplete markets \( l^i \) solves

\[
U_l[\Omega + (1 - l^i), l^i] = U_c[\Omega + (1 - l^i), l^i]. \tag{20}
\]

But we have already established in lemma 1 that \( Q > 0 \) in an interior solution with \( l < 1 \) provided that leisure is a normal good. Therefore, comparing equation (20) with (19), we observe the incomplete and complete market labor supply decisions differ only in that consumers who do work are richer under incomplete markets than under complete markets, since \( \Omega > \Omega - (1-\phi)Q! \)

Hence we conclude that, if leisure is a normal good, agents who work consume more leisure under incomplete markets than under complete markets (\( l^i > l \)). In other terms, labor supply is lower under incomplete markets than under complete markets.

The Dynamic Programming Problem of an Agent: Fix \( (w,1+r) > 0 \) with \( \beta(1+r) \leq 1 \) and let the mapping \( T \) on \( C(X) \), the space of bounded continuous functions on \( X \), be
as follows:

\[(TW)(x; w, r) = \sup_{(c, k') \in \Gamma(x; w, r)} \{ \mathcal{U}(x, x') + \beta E[W(x'; w, r)|x] \}, \tag{21} \]

and use the mapping \( T \) to define a sequence \( T^n \), with \( T^1 W = TW, T^2 W = T(TW), \ldots \) Then under A1-A4 and the constraints in Eq. (15), it can be shown that: (a) the operator \( T \) has a fixed point which is the unique solution to Eq. (13), and (b), there exist optimal decision rules for consumption \( c(x; w, r) \), leisure \( l(x; w, r) \) and capital holdings \( k(x; w, r) \) that attain the value function, i.e., the fixed point in \( T \) (see respectively Theorem 9.6 in Stokey and Lucas (1989) and Corollary 2 in Denardo (1967)).

Proof of Remarks: Proofs for \( R1 \) to \( R3 \) can be found in Huggett 1993 (pag. 964-68) and 1997 (pag. 399-400). For \( R4 \) we use the following FONC with respect to capital of the agent’s problem:

\[ u'(c(x, t)) + \mu(x, t) \geq \beta(1 + r)E[u'(c(k(x, t), s', t + 1))|x], \]

which holds with equality whenever \( k(x, t) > B \). In the previous condition \( \mu(x, t) \) is the multiplier associated to the upper bound for the stock of capital. This multiplier is equal to zero whenever \( k(x, t) < \bar{B} \). Notice that if \( B = 0 \) A2 implies that the condition above holds always with equality. For \( r = 0 \) the previous condition holds with equality for all \( B \geq 0 \). We also use the FONC with respect to leisure: \( l(k, 1) = b(w)c(k, 1) \). The decision of leisure has been simplified in the agent’s problem, but it must be satisfied.

\( R4 \): We consider different cases depending on the interest rate being strictly positive or not.

\( r > 0 \): Following the proof of Theorem 1 in Huggett (1993), under A1-A4 and \( k \in [B, \bar{B}] \) it can be shown by induction that \( v'(k, 1) \leq v'(k, 0) \), i.e., \( c(k, 0) \leq c(k, 1) \). Take \( k = \bar{k}_+ \), where \( \bar{k}_+ = (b(w)r)^{-1} \). For an agent in the unemployment state, \( R3 \) and the budget constraint implies that \( c(\bar{k}_+, 0) \geq b(w)^{-1} \). Since \( c(k, 0) \leq c(k, 1) \), then the FONC with respect to leisure implies \( l(\bar{k}_+, 1) = 1 \). Therefore \( k(\bar{k}_+, 1) \leq \bar{k}_+ \). This also means that we can fix \( \bar{B} = \bar{k}_+ \) without loss of generality.

\( r \leq 0 \): Take \( k_1 < k_2 \), thus \( c(k_1, 1) < c(k_2, 1) \). The budget constraint of an employed agent implies that:

\[ w(1 - l(k_1, 1)) + (1 + r)k_1 - k(k_1, 1) < w(1 - l(k_2, 1)) + (1 + r)k_2 - k(k_2, 1), \]

thus

\[ k(k_2, 1) - k(k_1, 1) < (1 + r)(k_2 - k_1) + w(l(k_1, 1) - l(k_2, 1)). \]
Since leisure is also strictly increasing in the level of capital, it follows that
\[
(k(k_2, 1) - k(k_1, 1))/(k_2 - k_1) < 1,
\]
and the proof is concluded.

An implication of the previous result is that capital accumulation can only continue as long as the agent’s labor supply is strictly positive. Thus one can choose without loss of generality an initial condition for capital and an upper bound \( \bar{B} \) such that both the multiplier associated to the time endowment and the multiplier associated to the upper bound for capital are always zero. In what follows we assume that this is the case.

**Proof of Proposition 2:** If \( \bar{B} > 0 \) and there exists a stationary equilibrium then consumption is bounded away from zero and Theorem 1 in Huggett 1997, pag. 391 provides the result. When \( \bar{B} = 0 \) the Euler Equation holds always with equality:
\[
u'(c(x, t)) = \beta f_K E[u'(c(k(x), s', t + 1))|x],
\]
Suppose, by way of contradiction, that \( \beta f_K = \beta(1 + r) = 1 \). Then Jensen’s Inequality implies that
\[
u'(c(x, t)) \geq u'(E[c(k(x), s', t + 1)|x]),
\]
and therefore, \( c(x, t) \leq E[c(k(x), s', t + 1)|x] \). Suppose that for a positive mass of agents the previous expression holds with strict inequality. Integrate both sides of the previous expression using the stationary distribution \( \psi \). Then Theorem 8.3 in Stokey and Lucas implies that:
\[
\int_X c(x, t) d\psi < \int_X E[c(k(x), s', t + 1)|x] d\psi = \int_X c(x, t + 1) d\psi^* = \int_X c(x, t + 1) d\psi,
\]
where the last step uses the fact that \( \psi^*(B) = \int P(s, B) d\psi \) for all \( B \in \mathcal{S} \) (see Theorem 8.3 in Stokey and Lucas (1989)), hence at the steady state \( \psi(B) = \psi^*(B) \). This is not possible, since at the steady state aggregate consumption is constant. Therefore we have that \( c(x, t) = E[c(k(x), s', t + 1)|x] \) \( \psi \)-everywhere. It is straightforward to show that for all \( x \) in \( X \):
\[
E[c(k(x), s', t + 1)|x] \geq E[c(0, s', t + 1)|x] > \pi_{1|1} c(0, 1) > 0. \tag{22}
\]
Thus if \( \beta f_K = 1 \) consumption is bounded away from zero. In fact, since \( \pi_{1|0} > 0 \) the arguments in Chamberlain and Wilson (Theorem 1, 2000) can be applied without change to show that marginal utility is a finite non negative martingale and thus, it converges. This means that the stock of capital also converges. Now, one possibility
is that \( k \) converges to a random variable such that the supremum of its support \( \hat{k} \) is strictly smaller than \( \bar{k}_+ \). Then \( l(\hat{k}, 1) < 1 \) and by R3 agents in the unemployment state decrease their capital stock. Since \( \pi_{0|0} \) and \( \pi_{0|1} \) are strictly positive, then at the steady state there would be a positive mass of agents with a capital stock arbitrarily close to zero (Lemma 1 in Huggett (1997)). Since consumption is strictly increasing in the level of capital this fact contradicts Eq. (22). Thus \( \bar{k}_+ \) must be in the support of the distribution of capital. But then the stock of capital converges to \( \bar{k}_+ \). Therefore at the steady state \( l(\bar{k}_+, 1) = 1 \) and no agent supplies any labor. The proof is concluded noticing that with labor supply equal to zero the interest rate would be equal to \(-\delta\), contradicting the hypothesis.
Figure 1: Decision rules for $s=1$ (thin line) and for $s=0$ (thick line) when the interest rate equals the rate of time preference.

Figure 2: Equilibrium decision rules for capital accumulation under incomplete markets.
Figure 3: Equilibrium distribution.