Lecture 4
Growth Model with Endogenous Savings:
Diamond Model

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The representative household assumption of the neoclassical growth model is not apt for the analysis of an economy in which there is turnover in population i.e. new households arrive (or are born) and old households leave (or die). Clearly, the notion of young and old individuals maps naturally into the real world. More importantly, this leads to new economic interactions absent in the neoclassical model. Specifically, the decisions made by old households affect the prices faced younger generations. To capture such interactions, in this chapter, we adopt the overlapping generations (OLG) framework which was introduced by Paul Samuelson and later studied by Peter Diamond.

1 Model Economy

Often, though not always, OLG models are better understood in a discrete time setup. So, we take time as discrete, \( t = 1, 2, \ldots, \infty \). Each individual lives for two periods. For now, let us start with a general separable utility function for individuals born at date \( t \):

\[
U(C_1(t), C_2(t + 1)) = u(C_1(t)) + \beta u(C_2(t + 1)) ,
\]

where \( u : \mathbb{R}_+ \to \mathbb{R} \) is strictly increasing, concave and twice differentiable, \( C_1(t) \) is consumption of an individual born at date \( t \) when young and \( C_2(t + 1) \) is this individuals consumption when old (at date \( t + 1 \)). \( \beta \in (0, 1) \) is the discount factor. We do not distinguish between individuals of the same age.

Factor markets are competitive. Individuals can work only when they are young, and they supply one unit of labor inelastically, earning wage \( w(t) \). Suppose, the size of generation \( t \) (individuals born at date \( t \)) is

\[
L(t) = (1 + n)^t L(0) .
\]

The production of the economy is the same as before. There are a large number of firms, each with production function

\[
Y(t) = F(K(t), A(t)L(t)) .
\]

The production function is constant returns to scale and satisfies Assumption 1 and Assumption 2, described in the Solow growth model. To keep things simple, we assume that capital depreciates fully, i.e. \( \delta = 1 \).
1.1 Firms’ Problem

Firms maximize their profit by choosing labor and capital. Taking $A(t)L(t)$ as effective labor we can write quantities per unit of effective labor. So $k = K/AL$ and $y = Y/AL = f(k)$. The first-order conditions give us the factor prices.

\begin{align*}
R(t) &= f'(k(t)) , \\
w(t) &= f(k(t)) - f'(k(t))k(t) .
\end{align*}

With full depreciation, $1 + r(t) = R(t)$.

1.2 Households’ Problem

Savings by an individual of generation $t$, $S(t)$, are pinned down by the solution to the following maximization problem:

\[
\max_{\{C_1(t), C_2(t+1), S(t)\}} \quad u(C_1(t)) + \beta u(C_2(t + 1)) , \\
\text{s.t.} \quad C_1(t) + S(t) \leq A(t)w(t) , \\
\text{and} \quad C_2(t + 1) \leq R(t + 1)S(t) ,
\]

where we are using the convention that young individuals rent their savings as capital to the firms at the end of period $t$ and receive the return at date $t + 1$ after production is carried out. Since $u(\cdot)$ is increasing, both constraints hold with equality. Therefore, the first-order conditions give us the consumption Euler equation.

\[
u'(C_1(t)) = \beta R(t + 1)u'(C_2(t + 1)) .
\]

Combining this condition with the lifetime version of the budget constraint, written as

\[
C_1(t) + \frac{C_2(t+1)}{R(t+1)} = A(t)w(t) ,
\]

gives us the the implicit function that determines savings of an individual.

\[
S(t) = S(A(t)w(t), R(t + 1)) .
\]

Since capital depreciates fully, and all new savings are invested in capital, the law of motion for capital is given by:

\[
K(t + 1) = L(t)S(t) = L(t)S((A(t)w(t), R(t + 1)) .
\]
2 Equilibrium

A competitive equilibrium is represented by sequences of aggregate capital stocks, household consumption, and factor prices, \(\{K(t), C_1(t), C_2(t), R(t), w(t)\}_{t=0}^\infty\), such that the factor price sequence \(\{R(t), w(t)\}_{t=0}^\infty\) is given by Eq(1) and Eq(2), individual consumption decisions \(\{C_1(t), C_2(t)\}_{t=0}^\infty\) are given by Eq(3), and the aggregate capital stock \(\{K(t)\}_{t=0}^\infty\) evolves according to Eq(5).

2.1 Characterizing the Equilibrium and the Steady-State

Divide both sides of Eq(5) \(A(t+1)L(t+1)\) and use the relationships \(L(t+1) = (1+n)L(t)\) and \(A(t+1) = (1+n)A(t)\) to express the law of motion of aggregate capital in terms of effective labor units:

\[
k(t + 1) = s\frac{(w(t), R(t + 1))}{(1 + n)(1 + g)} ,
\]

where \(s(\cdot, \cdot)\) is the savings in units of effective labor. Now substituting for factor prices from Eq(1) and Eq(2) gives us the fundamental law of motion in the OLG economy.

\[
k(t + 1) = \frac{s(f(k(t)) - f'(k(t))k(t), f'(k(t + 1))}{(1 + n)(1 + g)} . \tag{6}
\]

Then a steady state is given by a solution to this equation such that \(k(t + 1) = k(t) = k^*\).

\[
k^* = \frac{s(f(k^*) - f'(k^*)k^*, f'(k^*))}{(1 + n)(1 + g)} . \tag{7}
\]

Since the savings function \(s(\cdot, \cdot)\) can take any form, Eq(6) can lead to complicated dynamics and multiple steady states. The figure below illustrates this point. It shows that the OLG model can lead to a unique stable equilibrium, to multiple equilibria, or to an equilibrium with zero capital stock. In order to avoid these problems, we need to put some more structure on the utility function and the production function.

2.1.1 Additional Restrictions on Utility and Production Functions

Let us assume that the utility function takes the CRRA form:

\[
U(C_1(t), C_2(t + 1)) = \frac{C_1(t)^{1-\theta} - 1}{1-\theta} + \beta \left( \frac{C_2(t + 1)^{1-\theta} - 1}{1-\theta} \right) ,
\]

where \(\theta > 0\). Furthermore, let us assume that the production function is Cobb-Douglas:

\[
Y(t) = K(t)^{\alpha} (A(t)L(t))^{1-\alpha} \Rightarrow y = f(k) = k^\alpha .
\]
The CRRA utility function simplifies consumption Euler equation to

$$\frac{C_2(t+1)}{C_1(t)} = (\beta R(t+1))^{1/\theta}.$$ 

Substituting this relationship in the lifetime budget constraint gives us the expressions for consumption of the young and the savings in units of effective labor.

$$c_1(t) = \frac{\psi(t+1) - 1}{\psi(t+1)} w(t),$$

$$s(t) = w(t) - c_1(t) = \frac{1}{\psi(t+1)} w(t), \quad (8)$$

where

$$\psi(t+1) = 1 + \beta^{-1/\theta} R(t+1)^{-(1-\theta)/\theta} > 1.$$ 

c = CL/AL and s = SL/AL are consumption and savings expressed in effective labor units. Then

$$s_w = \frac{\partial s}{\partial w(t)} = \frac{1}{\psi(t+1)} \in (0, 1), \quad (9)$$

$$s_R = \frac{\partial s}{\partial R(t+1)} = \left(1 - \frac{\theta}{\theta} \right) (\beta R(t+1))^{-1/\theta} \frac{s(t)}{\psi(t+1)}.$$ 

So, 0 < s_w < 1, and s_R < 0 if $\theta > 1$, s_R > 0 if $\theta < 1$ and s_R = 0 if $\theta = 1$. Note that R affects the price of consumption when young with respect to consumption when old. Intuitively, an increase in

Figure 1: Multiple steady states in the general OLG model
leads to a substitution and income effect. An increase in $R$ makes consumption when young more expensive relative to consumption when old. This is the substitution effect. Plus, a higher $R$ means that a given amount of savings yield more consumption when old, which tends to decrease saving. This is the income effect. When individuals are very willing to substitute consumption between the two periods, i.e. $\theta < 1$, the substitution effect dominates which increases savings in response to an increase in $R$. On the other hand, when individuals are not very willing to substitute consumption between the two periods, i.e. $\theta > 1$, the income effect dominates which reduces savings in response to an increase in $R$. In the special case when $\theta = 1$ (logarithmic utility), the two effects balance and therefore the savings are independent of changes in $R$.

Given the expression for savings, in Eq(8), the law of motion for capital given by Eq(6) gives us the equation governing the dynamics of capital in this model.

$$k(t + 1) = \frac{w(t)}{(1 + n)(1 + g)} .$$

With a Cobb-Douglas production function, $R(t + 1) = \alpha k(t + 1)^{\alpha - 1}$ and $w(t) = (1 - \alpha)k(t)^\alpha$. Therefore, the dynamic equation now becomes

$$k(t + 1) = \frac{(1 - \alpha)k(t)^\alpha}{(1 + n)(1 + g)\left[1 + \beta^{-1/\theta} (\alpha k(t + 1)^{\alpha-1})^{-(1-\theta)/\theta}\right]} ,$$

which implies that the steady state capital stock $k(t + 1) = k(t) = k^*$, is given by the solution to the following equation.

$$(1 + n)(1 + g)\left[1 + \beta^{-1/\theta} (\alpha (k^*)^{\alpha-1})^{-(1-\theta)/\theta}\right] = (1 - \alpha) (k^*)^{\alpha-1} .$$

The equation above has a unique solution and this steady state value of capital is globally stable. Thus, starting with any $k(0) > 0$ and $k(0) \neq k^*$, the capital-labor ratio converges to the steady state value of $k^*$. This is depicted in the figure below.

3 The Standard OLG Model

Most OLG models use an even more specific form of the utility function - log preferences. Log utility function is a special case of the CRRA utility function, when $\theta = 1$. With log preferences, as discussed earlier, the substitution effect and the income effect cancel each other, so that changes in interest rate (or equivalently in the capital-labor ratio of the economy) have no effect on the saving
Figure 2: Equilibrium dynamics in the standard OLG model

rate. This result makes the equilibrium structure of the OLG model identical to that of the Solow model.

So, now the utility of an individual of generation $t$ is given by

$$U(C_1(t), C_2(t + 1)) = \log C_1(t) + \beta \log C_2(t + 1),$$

which implies that the consumption Euler equation is

$$\frac{C_2(t + 1)}{C_1(t)} = \beta R(t + 1),$$

which when substituted in the budget constraint yields the following expression for savings per unit of effective labor:

$$s(t) = \frac{\beta}{1 + \beta} w(t). \quad (12)$$

As a result, the dynamic equation for capital per unit of effective labor is

$$k(t + 1) = \frac{\beta w(t)}{(1 + n)(1 + g)(1 + \beta)},$$

which together with the Cobb-Douglas production function takes the following form:

$$k(t + 1) = \frac{\beta(1 - \alpha)k(t)^\alpha}{(1 + n)(1 + g)(1 + \beta)}. \quad (13)$$
And, therefore the steady state level of capital stock is given by:

\[ k^* = \left[ \frac{\beta(1 - \alpha)}{(1 + n)(1 + g)(1 + \beta)} \right]^\frac{1}{1-n} . \]  

(14)

The properties of the economy once it has converged to the balanced growth path (or steady state) are the same as those of the Solow and the Ramsey-Cass-Koopmans economies on their balanced growth paths: the savings rate is constant, output per worker and capital per worker are growing at rate \( g \), and therefore capital-output ratio is constant.

The response of this OLG economy to an increase in \( \beta \) (or fall in discount rate \( \rho \) because \( \beta = 1/(1 + \rho) \)) is similar to that of the Ramsey-Cass-Koopmans economy, and also to the response of the Solow economy to an increase in the saving rate \( s \). The change shifts the paths over time of output per worker and capital per worker up, but it leads only to temporary increases in the growth rates of these variables. This is depicted in the figure below. Higher \( \beta \) results in higher weight to future consumption, which causes higher savings. The \( k(t + 1) \) function shifts up, resulting in a higher steady capital stock.

![Figure 3: The effects of a fall in the discount rate](image)

4 Dynamic Inefficiency

There is one major difference between the balanced growth paths of the Diamond and Ramsey-Cass-Koopmans models - the implications for welfare. In the Ramsey-Cass-Koopmans model the
equilibrium allocations maximized individual’s welfare. However, in the Diamond model individuals of different generations attain different welfare levels. As a result it is not clear how should one evaluate social welfare. One could take a weighted some of the utilities of different generations, but then the question is what should be the weights?

One way to think about it would be ask ourselves if the equilibrium is *pareto efficient*. And, it turns out that the Diamond model need not satisfy the criterion of pareto efficiency. Specifically, the steady-state capital stock of the Diamond model may exceed the golden-rule level of capital stock (the capital stock that maximizes consumption). In such a case a permanent increase in consumption is possible.

To see this, let us take the standard OLG model with log utility, Cobb-Douglas production technology, and also assume that \( g = 0 \). Then, the steady-state capital stock is given by

\[
k^* = \left[ \frac{\beta(1-\alpha)}{(1+n)(1+\beta)} \right]^{\frac{1}{\alpha}},
\]

which implies that the marginal product of capital, \( R = f'(k^*) = \alpha (k^*)^{\alpha-1} \), is

\[
R^* = f'(k^*) = \frac{\alpha}{1-\alpha} \left( 1 + \frac{1}{\beta} \right) (1 + n),
\]

The golden-rule capital stock is the capital stock that maximizes consumption. Notice, that in the steady state of the OLG economy, we have

\[
f(k^*) - (1+n)k^* = c^*_1 + \frac{c^*_2}{1+n} \equiv c^*,
\]

which is saying that output minus investment is equal to value of lifetime consumption. \( c^* \) is the total steady-state consumption. Therefore,

\[
\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n),
\]

and the golden-rule level of capital stock \( K_{gold} \) is defined by

\[
f'(k_{gold}) = (1+n),
\]

which, given the Cobb-Douglas production function, implies that

\[
k_{gold} = \left[ \frac{\alpha}{(1+n)} \right]^{\frac{1}{\alpha-1}}.
\]

Therefore \( k^* \) may not be equal to \( k_{gold} \). If \( k^* > k_{gold} \), then \( \partial c^*/\partial k^* < 0 \), so reducing savings can increase the total steady-state consumption. If this is the case then the economy is said to
be dynamically inefficient. Because, by reducing savings one can increase total steady-state consumption, the equilibrium of the Diamond model is not pareto efficient. Another way of expressing dynamic inefficiency is $r^* < n$, i.e. the steady-state net interest rate ($r^* = R^* - 1$) is less than the rate of population growth.

This result may seem odd given that markets are competitive and there are no externalities. Recall, that in the Ramsey-Cass-Koopmans economy the transversality condition ensures that $r^* > g + n$. Therefore, dynamic inefficiency would never arise. In the Diamond model, the household heterogeneity associated with birth and death of household causes the inefficiency. Individuals from generation $t$ face wages determined by capital stock decisions of individuals from generation $t - 1$. Similarly, an individual from generation $t - 1$ receives a rate of return on her savings determined by savings decisions of others from generation $t - 1$. Thus, the savings decisions of each generation create pecuniary externalities on both workers and capital holders next date. These pecuniary externalities are the source of the inefficiency. Note, that pecuniary externalities are always present, but are of second order in competitive economies. In the OLG framework, however, pecuniary externalities need not cancel out when there is an infinite stream of newborn agents joining the economy.

Another way to think about dynamic inefficiency is that it arises from overaccumulation, which in turn arises from the need of the current young generation to save for old age. However, the more they save the lower the return to capital, and this may encourage them to save even more. This is the pecuniary externality that the decision of the young to save today imposes on future return to capital. If there were alternative ways of providing consumption to individuals in the old age, the overaccumulation problem might be solved or reduced. This is where social security comes in.