Instituto Tecnológico Autónomo de México

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*A discrete-bid first-price sealed-bid auction with complete information*

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Two bidders, 1 and 2, compete in order to get an object using the following auction mechanism. Each bidder will confidentially write her bid in a piece of paper and give this to the auctioneer. The auctioneer will then award the object to the higher bidder, who will have to pay the amount she bid (the "first price") for it. In case of a tie, the auctioneer flips a fair coin to decide who gets the object.

Let $v_i$ denote the maximum amount that bidder $i$, $i \in \{1,2\}$, is willing to pay in order to get the object (the bidder's value for the object). If the respective bids are $(b_1, b_2)$, then the utility of bidder $i$, taking $j \neq i$, is:

$$u_i(b_1,b_2) = \begin{cases} v_i - b_i, & \text{if } b_i > b_j; \\ \frac{1}{2}(v_i - b_i), & \text{if } b_i = b_j; \\ 0, & \text{if } b_i < b_j. \end{cases}$$

We will assume that both values and bids must be integer numbers between 0 and 100:

$$v_1, v_2, b_1, b_2 \in \{0, 1, 2, \ldots, 99, 100\}.$$

The values are assumed to be common knowledge. In order to find all Nash equilibria, we will first characterize the *best response* functions of the players. Taking again $i \neq j$, we have that:

$$\mathrm{BR}_i(b_j) = \begin{cases} \{b_i : b_i \leq b_j - 1\}, & \text{if } b_j \geq v_i + 1; \\ \{b_i : b_i \leq b_j\}, & \text{if } b_j = v_i; \\ \{b_j\}, & \text{if } b_j = v_i - 1; \\ \{b_j, b_j + 1\}, & \text{if } b_j = v_i - 2; \\ \{b_j + 1\}, & \text{if } b_j \leq v_i - 3. \end{cases} = \begin{cases} \{0, 1, \ldots, b_j - 1\}, & \text{if } b_j \geq v_i + 1; \\ \{0, 1, \ldots, b_j\}, & \text{if } b_j = v_i; \\ \{b_j\}, & \text{if } b_j = v_i - 1; \\ \{b_j, b_j + 1\}, & \text{if } b_j = v_i - 2; \\ \{b_j + 1\}, & \text{if } b_j \leq v_i - 3. \end{cases}$$

From here on, we will take $v_1 = 60$ and $v_2 = 50$. From the best reply, we may see that $b_i \in \mathrm{BR}_i(b_j)$ implies that $b_i \leq b_j + 1$, and also $b_i \leq \max\{v_i, b_j - 1\}$. If $(b_1, b_2)$ is a Nash equilibrium, then either $b_2 \leq v_2 = 50 \leq 57 = v_1 - 3$, in which case $b_1 = b_2 + 1$, or $b_2 \leq b_1 - 1$; in either case, it is player 1 who gets the object. That is, there are no (pure strategy) Nash equilibria in which player 2 gets the object.

We can distinguish two types of Nash equilibria:

- *Equilibria in which $b_i \leq v_i$ for both players*: In this case, since $b_2 \leq v_2 = 50 \leq 57 = v_1 - 3$, the best reply of player 1 implies $b_1 = b_2 + 1$. In order for $b_2$ to be a best reply by player 2, we must have $b_1 \geq 50 = v_2$. There are two equilibria satisfying both conditions: $(b_1, b_2) = (50, 49)$ and $(b_1, b_2) = (51, 50)$.
- *Equilibria in which $b_2 > v_2$*: In this case, we know that $b_2 \leq b_1 - 1$, ie $b_1 \geq b_2 + 1$. In order for this to be a best reply by player 1, it must be the case that $b_2 \leq v_1 - 2 = 58$. Concluding, any $(b_1, b_2)$ satisfying $51 \leq b_2 \leq 58$ and $b_1 = b_2 + 1$ fall into this category. For example, $(b_1, b_2) = (59, 58)$ or $(b_1, b_2) = (52, 51)$.

**Exercise.** Find all Nash equilibria (in pure strategies) when $v_1 = v_2 = 60$.