Bayesian Games

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Games with Incomplete Information

- A Game with Incomplete Information is a game in which the players have asymmetric information regarding some fundamental aspect of the game (players, strategies, or payoffs).
- In general, in a game with incomplete information there are several possible strategic (or extensive) forms, and the information of the players regarding which is the game they are actually playing is different for different players (ie, it is asymmetric).
- Note that, if players have differential information regarding any other aspect of the game that determines which strategic (extensive) form they are playing, this can be reduced to one of the fundamentals.

Games with Incomplete Information

- The most common type of such games is when the asymmetric information affects the payoffs of (some of) the players.
- We say that one aspect of the game is private information of one particular player when this player knows it with certainty, but nobody else does.
- Thus, in many games with incomplete information the payoff of one or more players is private information of the respective players.
- Note: In a more general definition, we might view a game as belonging to this category if some of the fundamentals of the game are not common knowledge among the players.

Battle of the Sexes with Incomplete Information

$2F$

		B_2	O_2
1	B_1	$3,1$	$0,0$
	O_1	$0,0$	$1,3$

$2U$

		B_2	O_2
1	B_1	$3,0$	$0,3$
	O_1	$0,1$	$1,0$

- In the battle of the sexes, both players decide independently whether to go to the ballet (B_i) or to the opera (O_i): Player 1 (she) prefers B_1 over O_1 and player 2 (he) has the opposite preference. But both prefer to go together than on their own.
- Consider the following variant of the game, player 2 is either faithful ($2F$) or unfaithful ($2U$): In the former case, he has the preferences of the standard game; in the latter, he prefers to be without the other player rather than with her (presumably because if she is not present he can go with a lover).
- If the fact that player 2 is faithful or not is private information of this player, then we are faced with a game with incomplete information.

Bayesian Games

- In principle, a game with incomplete information cannot be analyzed with the tools of Game Theory we have developed so far in the course.
- John C. Harsanyi proposed that we analyze such games with the standard tools by proceeding as follows:
  - Express the aspects of the game over which there is asymmetric information in terms of different possible types for each player.
  - Let types be private information of the respective players.
  - Assume that each player has a subjective probability distribution over the types of the remaining players. (This is what justifies calling such games Bayesian.)
- In general, it is assumed that there is a joint probability distribution over the types of all players which is commonly known. This is referred to as the common prior assumption. In this case, the first step in the Bayesian Game consists of the assignment by Nature of the types of all players according to their joint probability distribution.

Battle of the Sexes Example Continued

- In order to transform the Battle of the Sexes example into a Bayesian Game, we proceed as follows:
  - Suppose player 1 has a unique type (which we will continue to denote 1), and player 2 has two types: $2F$ (faithful) and $2U$ (unfaithful).
  - Suppose player 1 has a subjective probability distribution over the two types of the second player. For example, assume $2F$ has probability $2/3$ and $2U$ probability $1/3$.
- We have now two different ways of obtaining a Bayesian game, depending on whether we treat player 2 as a single player, or we assume that each type of player 2 is a different player.

Ex-Ante and Ex-Post Representations

- The Ex-Ante Representation of the Bayesian game is obtained when the players of the game are the original players.
- In this representation, we make the ficticious assumption that each player has not yet been assigned one of her possible types, so her strategy choice is contingent upon the type she will later learn.
- The Ex-Post Representation of the Bayesian game is obtained when each type of each player is treated as a different player in the new game.
- This second representation corresponds also to a fiction, because, for each player that has more than one type, we consider what each type would choose, were she the actual type of the player. (We may think of this as parallel universes, each of which is characterized by a particular assignment of types to players.)

Bayesian Equilibria

- A Bayesian Equilibrium is a Nash equilibrium either of the ex-ante or of the ex-post representation of the Bayesian game.
- We can leave this unspecified because there is a one-to-one correspondence between the pure-strategy Nash equiblibria of both representations.
- Actually, Kuhn's Theorem regarding the relationship between mixed and behavior strategies shows that there is also a correspondence (although not one-to-one) between mixed-strategy Nash equilibria of both representations.

Battle of the Sexes Example Continued

Player 1 (rows), Player 2 (columns):

	B_2B_2	B_2O_2	O_2B_2	O_2O_2
B_1	$3, 2/3$	$2, 5/3$	$1, 0$	$0, 1$
O_1	$0, 1/3$	$1/3, 0$	$2/3, 7/3$	$1, 2$

- This is the ex-ante representation of the variant of the Battle of the Sexes game. There are two players, 1 and 2.
- There is a unique pure-strategy equilibrium (B_1, B_2O_2), with equilibrium outcome $(2, 5/3)$.
- In the ex-post representation there are three players: 1, $2F$, and $2U$.
- The ex-post representation has a unique pure-strategy equilibrium (B_1, B_2, O_2), with equilibrium outcome $(2, 1, 3)$.
- The utility of player 2 in the equilibrium of the ex-ante representation is the expected value of the equilibrium utilities of players $2F$ and $2U$ in the ex-post representation.

Example with Types for Both Players

In this game, each player has two types:

1α vs 2γ:

	c	d
a	$20, 10$	$40, 30$
b	$50, 40$	$30, 20$

1α vs 2δ:

	c	d
a	$10, 0$	$20, 10$
b	$30, 20$	$40, 30$

1β vs 2γ:

	c	d
a	$60, 30$	$0, 40$
b	$40, 10$	$20, 50$

1β vs 2δ:

	c	d
a	$10, 20$	$20, 10$
b	$20, 30$	$10, 0$

There is a common prior, with joint distribution of types:

	2γ	2δ
1α	1/10	2/10
1β	3/10	4/10

Example with Types for Both Players

The ex-ante representation is:

Player 1 (rows), Player 2 (columns):

	cc	cd	dc	dd
aa	$26, 18$	$32, 16$	$10, 23$	$16, 21$
ab	$24, 16$	$22, 6$	$20, 30$	$18, 20$
ba	$33, 25$	$39, 23$	$13, 26$	$19, 24$
bb	$31, 23$	$29, 13$	$23, 33$	$21, 23$

- There is a unique pure-strategy Bayesian Nash equilibrium (bb, dc) with equilibrium outcome $(23, 33)$.
- We leave as an exercise for the reader to find the corresponding Bayesian Nash equilibrium of the ex-post representation and its equilibrium outcome.

Symmetric Uncertainty vs Asymmetric Information

Player 1 (rows), Player 2 (columns):

	c	d
a	$26, 18$	$16, 21$
b	$31, 23$	$21, 23$

- We have said that a game has incomplete information if there is asymmetric information among the players about which game they are facing.
- When there is symmetric uncertainty regarding which game the players face (ie, they are all equally uncertain and agree upon the respective probabilities), then they simply face the expected value of the possible games.
- For instance, suppose that in the previous example we assume that players know neither the own nor the opponent's type. Then they are facing the above (complete information) game.