Instituto Tecnológico Autónomo de México
Maestría en Finanzas

Economía Financiera (Eco-44105), 2015

An example of mixed-strategy Nash equilibrium

Ricard Torres

Each one of two individuals, 1 and 2, will confidentially write an integer number comprised between 1 and n, with $n \geq 2$, on a piece of paper. They will hand out those papers to a referee, who will read aloud the numbers written. If the two numbers coincide, then 2 will pay \$1 to 1, otherwise no money will be exchanged.

In this game, the respective strategies of the players will be integer numbers x and y, with $1 \leq x, y \leq n$. The utilities are:

$$u_1(x,y) = \begin{cases} 1, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases} \qquad u_2(x,y) = \begin{cases} -1, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Let $N = \{1, 2, \dots, n\}$, the set of possible strategies for both players. The best responses of both players are, for any $x, y \in N$:

$$B_1(y) = \{y\}, \qquad B_2(x) = N \setminus \{x\}.$$

Consequently, there is no pure-strategy Nash equilibrium.

Let $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ be the respective mixed strategies chosen by the players. We are going to derive the best response of player 1 to a mixed strategy q of player 2.

Assume first that there is j such that $q_j > q_i$, for any $i \neq j$. Consider first pure-strategy responses by player 1. If player 1 chooses a number x with probability 1, then her utility is going to be $u_1 = 1 \times q_x + 0 \times (1 - q_x) = q_x$. Therefore, the best pure-strategy response by player 1 is $x = j$, with which she obtains utility $u_1 = q_j$.

Consider now a mixed-strategy response p by player 1. The utility she gets is:

$$u_1(p,q) = p_1\, q_1 + p_2\, q_2 + \cdots + p_n\, q_n.$$

The reason is: with probability p_1, 1 will choose $x = 1$, in which case her utility will be q_1. With probability p_2 her choice will be $x = 2$ and her utility q_2. And so on.

Therefore the utility of 1 is going to be a convex combination of the numbers $\{q_1, q_2, \dots, q_n\}$. Since q_j is strictly larger than any of the other numbers, player 1 will attain the maximum utility when she chooses p with $p_j = 1$.

Concluding, when there is q_j that is strictly larger than any other q_i, player 1's best response is the pure strategy $x = j$.

This should not be surprising: The expected utility of player 1 given any mixed strategy choice q by player 2 is a *linear function* of the probability vector $(p_1, p_2, \dots, p_n)$. Therefore, the maximum is always attained at an extreme point, in which $p_i = 1$ for some i (and therefore 0 for all other components). In other words, for any mixed strategy q of player 2, player 1 will always have some pure strategy which is a best response (although we do not discard the existence of mixed strategies that are also best responses).

Exercise 1. Given any mixed strategy q of player 2, show that any best response p of player 1 satisfies: $p_j > 0$ implies that $q_j = \max\{q_1, q_2, \ldots, q_n\}$. Conversely, show that any such p is a best response to q.

Consider now best responses of player 2 to a mixed strategy p by player 1. Since the maximum utility player 2 can attain is 0, any strategy that yields this utility is a best response. If there is j such that $p_j = 0$, then the strategy q with $q_j = 1$ will give player 2 a utility 0 and is a best response to p.

Exercise 2. Given any mixed strategy p by player 1, show that any best response q of player 2 satisfies: $q_j > 0$ implies that $p_j = \min\{p_1, p_2, \ldots, p_n\}$. Conversely, show that any such q is a best response to p.

Exercise 3. Show that the game has a unique mixed-strategy Nash equilibrium in which $p_i = q_i = 1/n$ for any $i \in \{1, 2, \ldots, n\}$.