Defining States of the World

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States of the World

- In the Savagean (after Leonard Savage) framework for decision-making under uncertainty, the first step is the identification of all relevant sources of uncertainty for the problem one is considering.
- Next, we define a state of the world as a possible outcome of the resolution of all those uncertainties.
- The same definition is applied in problems of general equilibrium under uncertainty, in which one defines contingent plans (or contingent commodities) by using the states of the world as an indexing variable for the different goods, much in the same way as one indexes goods by time or location.

The "Monty Hall" example

- A contestant tries to choose a prize that has been placed behind one of three doors, A, B, and C, with equal probabilities.
- Initially, the contestant chooses one of the doors.
- Out of the two remaining doors, the host (Monty Hall) will open one that does not have the prize behind.
- Next, the contestant is asked whether she prefers to stick with her original choice or switch to the remaining door.

Usual reactions

- Let's suppose that the contestant originally chooses door A, and then the host opens door C.
- There are two ways of reasoning about this problem:
  1. The apparently naïve way: "My initial probabilities were 1/3 for A and 2/3 for $\{B, C\}$. Therefore, after C is opened my probabilities are 1/3 for A and 2/3 for B."
  2. The apparently sophisticated way: "Given that my initial estimates are that A and B have equal probability, after C is opened I have no reason to modify their relative weights, so the conditional probabilities of A and B must still be equal, ie 1/2 each."
- Given that economists tend to view themselves as sophisticated thinkers, most of them adhere to the second alternative on a first reaction.

Conditional probabilities and information

- One way to formally solve the dilemma consists of computing the conditional probabilities of A and B, given the information that has been received.
- The problem with the wrong way of reasoning (which is the apparently sophisticated one) is that the information received is misrepresented.
- If we interpret that the information received is "the prize is not behind door C," then the apparently sophisticated interpretation is right. For instance:

$$\text{Prob}(A|\text{Not } C) = \text{Prob}(A|\{A,B\}) = \frac{\text{Prob}(A\&\{A,B\})}{\text{Prob}(\{A,B\})} =$$

$$= \frac{\text{Prob}(A)}{\text{Prob}(\{A,B\})} = \frac{1/3}{2/3} = \frac{1}{2}$$

What exactly is the information received?

- However, the information received is not exactly that the prize is not behind door C.
- This would be the right way to represent the information if, before the contestant takes any decision, the host opens door C and shows that there is no prize behind.
- But here the host has opened door C after the contestant chose door A.
- This subtle distinction makes a huge difference regarding the computation of the conditional probabilities.

Resorting to states of the world

- A systematic approach in order to compute the right conditional probabilities is based on defining the states of the world for this problem.
- There are four sources of uncertainty: Where the prize is, and which door the host will open for each of the 3 possible initial choices of the contestant.
- So the state space has 4 dimensions.

Defining the states of the world

- The first dimension represents the location of the prize, and therefore it has 3 possibilities (A, B, or C).
- The second dimension represents the door the host will open if the contestant initially chooses door A, and therefore there are two possibilities (open B or C).
- Analogously, the third and four dimensions represent the door that the host will open whenever the initial choice of the contestant is, respectively, B or C.
- This makes a total of 24 possible states.

Description of the states of the world

- However, some of those 24 possible states have probability zero, given the rules of the game.
- These are those states in which the prize is behind a certain door and then the host opens this door: this can never happen.
- Taking this fact into account, we are left with just 6 states that have positive probability.
- If D represents one of the doors (A, B, or C), then let us denote with pD the location of the prize, and with oD the door that the host opens.
- Then the states that have positive probability are:

$$\begin{array}{cc} (pA, oB, oC, oB) & (pA, oC, oC, oB) \\ (pB, oC, oA, oA) & (pB, oC, oC, oA) \\ (pC, oB, oA, oA) & (pC, oB, oA, oB) \end{array}$$

Assigning probabilities

- Thus, for example, (pA, oB, oC, oB) means that the prize is behind door A, and the host will open B if A is initially chosen, C if B is initially chosen, and B if C is initially chosen.
- The assignment of probabilities depends on what the contestant estimates the strategy of the host will be.
- Let us assume, for the sake of simplicity, that the host will choose with equal probabilities whenever he has a choice of opening two doors, and these choices are independent among themselves and also independent of where the prize is.

Probabilities of the states

☞ With this assumption, the probabilities of the different states are multiplicative:

State	Probability
(pA, oB, oC, oB)	$\frac{1}{3} \times \frac{1}{2} \times 1 \times 1 = \frac{1}{6}$
(pA, oC, oC, oB)	$\frac{1}{3} \times \frac{1}{2} \times 1 \times 1 = \frac{1}{6}$
(pB, oC, oA, oA)	$\frac{1}{3} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{6}$
(pB, oC, oC, oA)	$\frac{1}{3} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{6}$
(pC, oB, oA, oA)	$\frac{1}{3} \times 1 \times 1 \times \frac{1}{2} = \frac{1}{6}$
(pC, oB, oA, oB)	$\frac{1}{3} \times 1 \times 1 \times \frac{1}{2} = \frac{1}{6}$

Conditional probabilities

- ✎ Probability of win if staying put: The conditional probability of the prize being behind A whenever the contestant has chosen A initially and the host has opened B is:

$$\frac{\text{Prob}(pA, oB, oC, oB)}{\text{Prob}(pA, oB, oC, oB) + \text{Prob}(pC, oB, oA, oA) + \text{Prob}(pC, oB, oA, oB)} =$$

$$\frac{1/6}{(1/6) + (1/6) + (1/6)} = \frac{1}{3}$$

- ✎ Probability of win if switching: The conditional probability of the prize being behind C whenever the contestant has chosen A initially and the host has opened B is:

$$\frac{\text{Prob}(pC, oB, oA, oA) + \text{Prob}(pC, oB, oA, oB)}{\text{Prob}(pA, oB, oC, oB) + \text{Prob}(pC, oB, oA, oA) + \text{Prob}(pC, oB, oA, oB)} =$$

$$\frac{(1/6) + (1/6)}{(1/6) + (1/6) + (1/6)} = \frac{2}{3}$$

Conclusion

- Therefore, whenever the contestant chooses initially A and the host opens B, the contestant should decide to switch to C.
- It is easy to see that, in all cases, switching to the remaining door will give *a* probability of winning equal to 2/3, while staying put will has an attached probability of 1/3.
- Actually, the apparently naïve reasoning we described before is right.

A different way of getting the right answer

- There is a very simple way of reasoning to obtain the right answer.
- Think in frequentist terms, assuming that the law of large numbers holds exactly.
- Suppose that the game is repeated 300 times, and that the prize is behind each door exactly 100 times.
- Then, if the contestant initially chooses A and next switches to the remaining door, she will win in the 200 cases in which the prize is either behind B or behind C, while maintaining the choice of A will only win in 100 cases.
- That is, the apparently naïve way of reasoning is right.