Instituto Tecnológico Autónomo de México

Maestría en Economía

Microeconomía Aplicada II, 2015

Reduction of Compound Lotteries and Random Variables

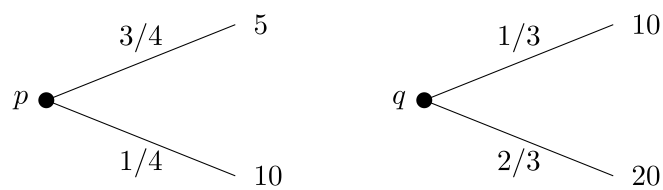
Ricard Torres

1 Compound lotteries

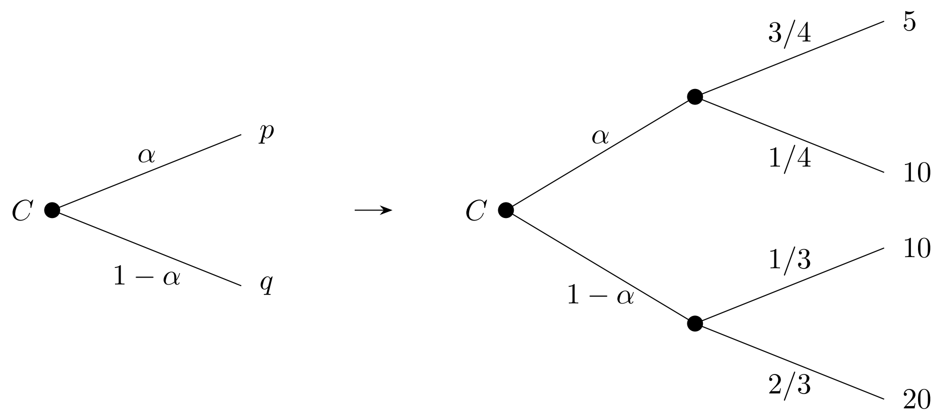
When considering decision making under risk, it is common to take as point of departure the set of all simple probability distributions on the real line (or on the nonnegative real numbers). In this case, a basic assumption in the derivation of the von Neumann-Morgenstern expected utility theory is the so-called *axiom of reduction of compound lotteries*. Here, lotteries are identified with (simple) probability distributions.

Each lottery is interpreted as the result of running a random device (say, a conveniently modified roulette wheel). A *compound lottery* consists of running a random device the results of which are not monetary prizes, but lotteries. For example, given two lotteries p and q, a compound lottery is the result of running first a device which will yield lottery p with a certain probability α, and lottery q with probability $1 - \alpha$.

Consider the following example. The two simple lotteries are:



And the compound lottery is:



Call E the random event which has probability α that is used to define the compound lottery. We should note that, in the compound lottery, the probabilities that correspond to the simple lotteries p

and q are now *conditional* on the occurrence or not of the event E. Taking this into account, we may easily compute the probabilities with which each of the possible outomes, $(5, 10, 20)$, may occur; these are, respectively, $(3\alpha/4, (4-\alpha)/12, (2/3)(1-\alpha))$. The axiom of reduction of compound lotteries states that the individual identifies (or is indifferent between) the compound lottery C and the simple lottery that corresponds to the above distribution over final outcomes. That is, the individual is indifferent with respect to *the way in which the uncertainty is generated*, and only cares about the final outcomes and their respective probabilities.

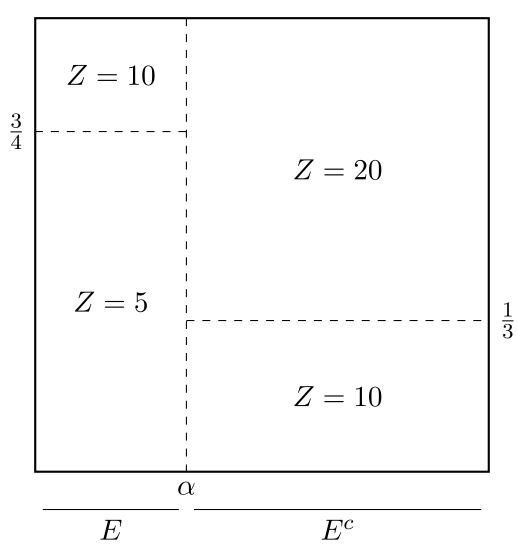
2 The random variable framework

Suppose now that, instead of identifying lotteries with probability distributions, we think of them as random variables. For example, suppose we consider random variables X and Y defined by: $X = 5\,I_{(0,3/4]} + 10\,I_{(3/4,1]}$ and $Y = 10\,I_{(0,1/3]} + 20\,I_{(1/3,1]}$. These two random variables have as respective distributions the p and q we used in the previous example.

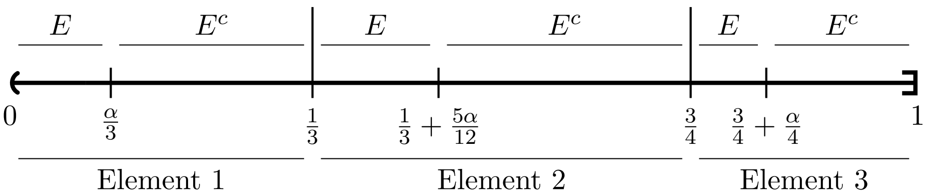
A mathematically natural way of defining a two-step uncertainty process similar to the one used in the compound lottery, consists of assuming that our probability space is now $(0,1] \times (0,1] = (0,1]^2$, with the uniform distribution on the unit square. The first component corresponds to the generation of the compound lottery part, and the second to the resolution of the individual random variables. We can easily generate the compound lottery distribution by taking $E = (0, \alpha]$, and considering a random variable $Z : (0,1]^2 \to \mathbb{R}$, defined on the unit square, with values:

$$Z(\omega_1, \omega_2) = \begin{cases} 5, & \text{if } \omega_1 \in (0,\alpha], \omega_2 \in (0, 3/4]; \\ 10, & \text{if } \omega_1 \in (0,\alpha], \omega_2 \in (3/4, 1]; \\ 10, & \text{if } \omega_1 \in (\alpha,1], \omega_2 \in (0, 1/3]; \\ 20, & \text{if } \omega_1 \in (\alpha,1], \omega_2 \in (1/3, 1]. \end{cases}$$

Note that, when $\omega_1 \in E = (0, \alpha]$, the conditional distribution of ω_2 is that of X, and when $\omega_1 \in E^c = (\alpha, 1]$ the conditional distribution of ω_2 is that of Y. So this random variable Z corresponds to the generation the compound lottery via a two-step process.



Now, we may generate another random variable Z' with just one source of uncertainty (ie, with probability space the unit interval, instead of the square), using the following process. Since both X and Y are given in partitional form, we consider first the *coarsest common refinement* of the two partitions (a *refinement* of a partition is the result of subpartitioning the elements of this partition; we are looking for a common refinement with the least number of elements). A simple procedure leads to this new partition: just list in sequential order the cutting points that define the two partitions. In our case, this results in $(0, 1/3, 3/4, 1)$: we have now a partition with three elements. Now, for each of the new elements (intervals), single out a subinterval which is of α times the length of the interval: this subinterval we will associate with E, and what is left of the interval we will associate with its complement.



On the element 1 of the partition, the interval $(0, 1/3]$, the respective values of X and Y are $(5, 10)$; on the element 2, the interval $(1/3, 3/4]$, the respective values are $(5, 20)$; finally, on the element 3, the interval $(3/4, 1]$, the respective values are $(10, 20)$. Let E be the union of the three subintervals indicated on the diagram:

$$E = \left(0, \frac{\alpha}{3}\right] \cup \left(\frac{1}{3}, \frac{1}{3} + \frac{5\alpha}{12}\right] \cup \left(\frac{3}{4}, \frac{3}{4} + \frac{\alpha}{4}\right]$$

Note that, by construction, E has length (probability) α. Moreover, the random variable I_E is independent of both X and Y because, by construction, $\mathbb{P}(I_E = 1|X = x) = \alpha = \mathbb{P}(E) = \mathbb{P}(I_E = 1)$ for $x \in \{5, 10\}$, and $\mathbb{P}(I_E = 1|Y = y) = \alpha$ for $y \in \{10, 20\}$.

Define now the random variable $Z' : (0, 1] \to \mathbb{R}$ by:

$$Z'(\omega) = X(\omega)\, I_E(\omega) + Y(\omega)\, I_{E^c}(\omega).$$

So that Z' coincides with X when $\omega \in E$, and with Y when $\omega \in E^c$. Now, it is easy to see that Z', which is generated by one-step uncertainty, gives rise to the same distribution Z, as well as the compound lottery C, generate on the real numbers:

$$\begin{aligned}
\mathbb{P}\{\omega \in (0,1] : Z'(\omega) = 5\} &= \mathbb{P}\left(0, \frac{\alpha}{3}\right] + \mathbb{P}\left(\frac{1}{3}, \frac{1}{3} + \frac{5\alpha}{12}\right] = \frac{3\alpha}{4} \\
\mathbb{P}\{\omega \in (0,1] : Z'(\omega) = 10\} &= \mathbb{P}\left(\frac{\alpha}{3}, \frac{1}{3}\right] + \mathbb{P}\left(\frac{3}{4}, \frac{3}{4} + \frac{\alpha}{4}\right] = \frac{4-\alpha}{12} \\
\mathbb{P}\{\omega \in (0,1] : Z'(\omega) = 20\} &= \mathbb{P}\left(\frac{1}{3} + \frac{5\alpha}{12}, \frac{3}{4}\right] + \mathbb{P}\left(\frac{3}{4} + \frac{\alpha}{4}, 1\right] = \frac{2}{3}(1-\alpha).
\end{aligned}$$

Concluding, in the framework based on random variables, the parallel of the axiom of reduction of compound lotteries is the statement that the individual identifies (or is indifferent between) the above random variables Z (with a two-dimensional state space) and Z' (with a one-dimensional state space).