LIQUIDITY DISCOVERY AND ASSET PRICING *

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Abstract

Asset prices are random, in part, because of uncertainty about the preferences of potential counterparties and their future demands for securities. We call such randomness liquidity risk. We model the endogenous dynamics of liquidity risk, the risk premium for bearing liquidity risk, and the role of market trading in the liquidity discovery process through which investors form expectations about future liquidity. Our model provides explanations for “price support levels” and “flights to quality”.

1 Introduction

Stocks and other long-dated assets are claims on streams of cash flows that continue long after the likely holding horizons of most investors. Although the value of long-dated securities derives ultimately from their cash flows, the anticipated preferences of potential trading counter-parties are also important determinants of investors’ current demands for long-dated assets. In particular, the counter-party preferences lead to the net demands for securities that investors will face if they need to re-trade in the near future.

The interaction of prices, order flows, and uncertainty about counter-party preferences is at the intersection of general equilibrium asset pricing and market microstructure. The canonical asset pricing model of Lucas [1978] assumes that investor trading strategies are common knowledge. Prices are random because investors’ strategies are contingent on the arrival of news about random cash flows, but there is no learning about counter-party preferences. The only learning is about the exogenous cash flows. Moreover, the trading process in such a model does not reveal information. It simply “digests” information that arrives exogenously from other sources by reallocated securities across investors.

Learning is central in the microstructure approach of Kyle [1985] and Glosten and Milgrom [1985] and in the rational expectations model of Grossman and Stiglitz [1980]. Since informed investors see non-public signals about future cash flows, the trading process itself is informative. However, uninformed investors only learn about cash flows but not about their informed counterparties’ preferences. Indeed, the informed investors’ preferences play no direct role in asset pricing by the risk neutral dealers in Kyle [1985] and Glosten and Milgrom [1985] once the signal extraction problem is solved.

Our starting point is the idea, first described in Grossman [1988] and Kraus and Smith [1989], that future prices are random because investors do not know the preferences of other investors. Such uncertainty is in addition to any uncertainty about future cash flows. We call the traders’ uncertainty about other traders’ preferences liquidity risk and the traders’ uncertainty about future cash flows cash flow risk.

It is natural to suppose that preferences are not common knowledge. First, investors’ preferences may be subject to random shocks over time. Indeed, it is possible that investors are not fully
informed a priori about the possible evolution of their own preferences in the future. In this case, neither the investor himself nor his counter-parties know what his demand for securities will be at future dates. It is reasonable, however, to suppose that investors learn their personal preferences before these preferences become known to others. Second, an investor may know his own future preferences a priori, but this information may not be common knowledge among his counter-parties. In other words, the investor’s preferences appear random to others, but not to himself.

Both intuitions are consistent with the analysis in our paper. In either case, counter-parties can learn about the investor’s likely demand for securities in the future from the demand the investor’s preferences induce for securities today. Thus, trading itself generates random price changes as in French and Roll [1986] — even without random cash flows. In such a circumstance, the trading process is both a mechanism for learning about the likely impact of non-public investor preferences on future prices as well as for digesting information about exogenous cash flows. We call the process of learning about counter-party preferences liquidity discovery because investors are uncertain a priori about the prices at which they will find willing counter-parties in the future. Liquidity discovery reduces liquidity risk.

We develop a model to illustrate how liquidity risk can arise and the specific relation between trading and liquidity discovery. Investors differ in their holding periods. Short-horizon investors trade default-free long-dated bonds with long-horizon investors, but are uncertain about the long-horizon investors’ time preferences. In particular, short-horizon investors’ can infer the long-term investors’ aggregate discount rate for the full period, but not necessarily the specific discount rates for each subperiod. Consequently the short-horizon investors are uncertain about the price at which they will be able to trade the bond in the future. Our main results are:

• Prices are random because of uncertainty about the long-lived investors’ future preferences.

• Liquidity risk is a priced risk factor. Expected bond returns include a liquidity premium in excess of the risk-free rate.

• Preference information is either fully or partially revealed via order flows. Thus, the level of prevailing liquidity risk is endogenous.

• The liquidity supply function exhibits endogenous price support levels. Small order flow
variations can lead to discontinuous price jumps (e.g., “crashes”) from one price support level to another.

- Trading volume is informative about variation in the magnitude of future liquidity risk and its current risk premium. Sufficient conditions are given for a non-monotone relation between prices, volumes, liquidity risk, and the liquidity risk premium.

- The predicted co-movements of volume, prices, returns and bond-bill spreads are consistent with so-called “flights to quality” such as in the aftermath of the 1998 Russian bond default.

- Pre-announcing investor preferences does not Pareto improve investors’ welfare. Unverifiable announcements of the long-horizon investors preferences are not incentive compatible and hence do not reduce liquidity risk.

In addition, the process of liquidity discovery offers a new explanation for a growing body of empirical evidence showing that traditional microstructure variables — such as price impact coefficients or the probability of informed trading — are “priced” in the sense that they explain expected returns. This relation was first documented in Amihud and Mendelson [1986]. Easley, Hvidjkaer, and O’Hara [2002] and Stambaugh and Pastor [2002] are two examples of recent work on this topic.

As markets clear over time, investors form expectations about the amount of market liquidity available from other investors. In particular, they are interested not only in current liquidity, but also in anticipated future liquidity. Imperfectly known future liquidity leads to technical support and resistance levels in current prices. Large market volume causes prices to break through the prevailing support and resistance level and leads to uncertainty about where the “points of liquidity” will be in the future. The resulting increased liquidity risk leads, in turn, to higher risk premia until the new support levels are discovered. In this context we can also interpret “liquidity crises” as subsequent future realizations of liquidity that are less than what was expected ex ante.

Our paper builds on the work by Grossman [1988] and Kraus and Smith [1989]. Grossman [1988] introduced the idea that investors’ future trading plans may not be common knowledge and that, consequently, prices may change, sometimes dramatically, as the flow of orders into the market reveals the latent strategies investors are following. In particular, market prices may reveal
information about investors’ current demands without fully revealing their future demand functions.

Kraus and Smith [1989] is the first formal model of liquidity risk. Their model has multiple “sunspot” equilibria. In one equilibrium trade occurs without resolving any of the uncertainty about counter-party preference. In another counter-party information is not revealed at all. One contribution of our paper is that we model both the extent to which counter-party information is revealed and the microstructure mechanism through which this happens. We also study the impact of liquidity risk on market risk premia.

Other related work includes Kraus and Smith [1996, 1998] which uses counter-party uncertainty to endogenize noise trading in a rational equilibrium model with asymmetric cash flow information. Smith [1993] develops an overlapping generations model in which liquidity risk takes the form of uncertainty about how many traders will arrive in the future. Since traders are risk averse, future prices are random because the aggregate future demand for intertemporal smoothing is random. Leach and Madhavan [1992] and Saar [2001] both model dynamic updating of market maker beliefs about investor demand based on observed order flows. Vayanos [1999, 2001] models strategic trading when a large investor needs to trade for non-informational reasons, but can choose the timing of his trades. Liquidity providers seek to learn his preferences — that is, the total number of shares he intends to buy — by observing his trades over time. Pedersen and Acharya [2003] provide an explanation for the Stambaugh and Pastor [2002] results in a model with cross-sectionally correlated transaction costs. In contrast, our approach links the concept of liquidity to the fundamental demand for securities rather than to transactional frictions.

Following the market crash of 1987, a number of papers sought to explain how small variation in order flow could lead to discontinuous changes in price. In Gennotte and Leland [1990] discontinuous price change arises in a rational expectation setting due to confusion about whether traders are informed about future payoffs or are informed about noisy supply shocks. Madrigal [1996] considers the impact of private information about noise trading in a Kyle [1985] model. Madrigal and Scheinkman [1997] has discontinuous price changes in a model of strategic market marker learning with heterogeneously informed agents. The key feature in each of these three papers is that, over some regions, small changes in trading volume can lead to large revisions in expectations about future cash flows. In contrast the abrupt price changes in our model arise due to uncertainty about
the endogenous prices at which current counter-parties will be willing to re-trade in the future.

Our paper is organized as follows. Section 2 presents a model of liquidity risk based on uncertainty about the time preferences of a long lived investor. Section 3 is the conclusion. All proofs are in the Appendix.

2 Liquidity risk and time preferences

We consider a pure exchange economy with three trading dates \( t = 1, 2, 3 \) with one traded security. A two-period discount bond pays consumption at time 3. The bond is default-free and so has no cash flow risk. Although we refer to a single long-dated bond, it is more accurate to think of our bond as proxying for the entire long-term bond market. Our model is a general equilibrium model of market-wide, or systematic, liquidity risk premia.

Two groups of competitive investors trade with each other. The first group is a continuum of long-horizon investors, denoted by the subscript \( L \), with three-period preferences

\[
\begin{aligned}
   u(c_{L,1}) + (\delta + \epsilon) u(c_{L,2}) + (\delta + \epsilon) \delta u(c_{L,3}).
\end{aligned}
\] (1)

The long-lived investors have individual endowments of \( e_{L,1} = 0 \) of consumption at date 1, \( e_{L,2} > 0 \) units of consumption at date 2, and start out holding all of a long-dated bond which pays 1 unit of consumption at date 3. The two time preference parameters \( \delta \) and \( \epsilon \) are random and are known ex ante only to the long-lived investors. For reasons that will become clear shortly, this implies there is no price randomness from the perspective of the long-lived investors. Hence, no expectations are necessary in (1).

The persistent component of the long-lived investors’ time preferences, \( \delta \), is binomially distributed. With probability \( \pi_I \) the long-lived investors are impatient and use a discount factor \( \delta = \delta_I \) which assigns a low value to future consumption. With probability \( \pi_P = 1 - \pi_I \), they are patient and used a discount factor \( \delta_P > \delta_I \) which values future consumption more highly. The second term \( \epsilon \) is a transitory shock which causes their time preference \( \delta + \epsilon \) between dates 1 and 2 to differ from their time preference \( \delta \) between dates 2 and 3. Let \( \epsilon_P \) denote the realized value of \( \epsilon \) conditional on a permanent preference \( \delta_P \) and let \( \epsilon_I \) denote the realization of \( \epsilon \) conditional on \( \delta_I \). The probability density for \( \epsilon_P \) is \( f_P \) and the density for \( \epsilon_I \) is \( f_I \). We assume \( \epsilon_P \) and \( \epsilon_I \)
have continuous distributions to keep the equilibrium from being generically revealing as in Radner [1972].

The second group is a continuum of identical short-horizon investors, denoted by the subscript $S$, who have expected utility preferences over consumption at dates 1 and 2:

$$u(c_{S,1}) + \beta E_{S,1}[u(c_{S,2})],$$

(2)

where $E_{S,1}$ denotes the short-lived investors’ expectations given the information available to them at date 1 and $u$ is increasing, concave, differentiable and satisfies the Inada conditions. The short-term investor’s preferences are common knowledge. The short-term investors have individual endowments of $e_{S,1} > 0$ units of consumption at time 1 and $e_{S,2} \geq 0$ unit of consumption at date 2.

Given the absence of any cash flow randomness, there are only two states. Either bond prices are high at time 2 because the long-lived investors are patient or bond prices are low because they are impatient. Since the long-lived investors know their preferences at time 1, they face no bond price risk at time 2.

In equilibrium, the short-lived investors learn about the long-lived investors’ time preferences from prices and trades. If trading at date 1 is fully revealing about the long-lived investor’s preferences, then the long-term bond is also riskless for the short-lived investors since the bond price at date 2 only depends on $\delta$. If $\delta$ cannot be inferred at time 1, then short-lived investors are uncertain about the bond price at date 2. In this case, owning the two-period bond exposes them to liquidity risk. Long-term investors, however, have no liquidity risk. From their perspective, the long-term bond is riskless between dates 1 and 2 as well as between 2 and 3.

**Trading.** The motive for trade in this economy is intertemporal consumption smoothing. If short-term investors cannot perfectly anticipate their trading counter-parties’ future security demands, then the resulting liquidity risk prevents them from smoothing their consumption perfectly.

Long-lived investors, given their zero consumption endowment $e_{L,1}$, sell some of their bonds to finance consumption $c_{L,1}$ at time 1. Later they use their endowment at date 2 to buy the bonds back. This allows them to shift consumption from date 2 to date 1. If $\theta_{L,1}$ denotes the number of
bonds the long-horizon investors hold at date 1, then $1 - \theta_{L,1}$ is the number of bonds they sell per capita to the short-lived investors. At time 2 the long-lived investors buy back $\theta_{L,2} - \theta_{L,1}$ bonds to bring their total holdings of the bond at time 2 to $\theta_{L,2}$. In equilibrium, they buy back all of the bonds since the short-lived investors do not value consumption at time 3 implying $\theta_{L,2} = 1$ and $\theta_{L,2} - \theta_{L,1} = 1 - \theta_{L,1}$.

**Notation.** Let $P_1 = P_1(\theta_{L,1})$ be the price of the two-period bond. It is a function of the long-horizon investors’ observable portfolio holdings $\theta_{L,1}$ at date 1. Let $P_2 = P_2(\delta, \theta_{L,1})$ denote the price of the two-period bond at time 2 as a function of both the long-horizon investors’ preference parameter $\delta$ and the amount $1 - \theta_{L,1}$ of long-dated bonds that they must buy back in equilibrium.

Short-lived investors shift consumption between date 1 and 2 by trading the bond with the long-lived investors. To increase consumption at time 2 they buy two-period bonds at time 1 and resell them at time 2 for either $P_2(\delta_P, \theta_{L,1})$ or for $P_2(\delta_I, \theta_{L,1})$.

A time line is in Figure 1. We also introduce a shadow price for a one-period riskless bill, denoted $b_1$. The shadow is from the short-lived investors’ perspective.

**The investor problems.** The long-term investor’s optimization problem is

$$
\max_{\theta_{L,1}, \theta_{L,2}} u(c_{L,1}) + (\delta + \epsilon) u(c_{L,2}) + (\delta + \epsilon) \delta u(c_{L,3}),
$$

subject to

$$
c_{L,1} = P_1(1 - \theta_{L,1}),
$$

$$
c_{L,2} = e_{L,2} - P_2(\theta_{L,2} - \theta_{L,1}),
$$

$$
c_{L,3} = \theta_{L,2}.
$$

The first-order conditions for the long-term investor’s optimal holdings of the two-period bond, given that they know their own $\delta$ and $\epsilon$ are

$$
P_1 = \frac{(\delta + \epsilon) u_c(e_{L,2} - P_2(\theta_{L,2} - \theta_{L,1}))P_2}{u_c(P_1(1 - \theta_{L,1}))},
$$

$$
P_2 = \frac{\delta u_c(\theta_{L,2})}{u_c(e_{L,2} - P_2(\theta_{L,2} - \theta_{L,1}))}.
$$
In equilibrium, the long-lived investors buy back all of the outstanding bonds at date 2 since the short-term investors have no demand for consumption at date 3. Imposing the market-clearing restriction, \( \theta_{L,2} = 1 \), and substituting (6) into (5), gives prices for the long-term bond

\[
\begin{align*}
P_1 &= \frac{(\delta + \epsilon) \delta u_c(1)}{u_c(P_1(1 - \theta_{L,1}))} \\
P_2 &= \frac{\delta u_c(1)}{u_c(e_{L,2} - P_2(1 - \theta_{L,1}))}.
\end{align*}
\]

The long-lived investor’s preferences \( \delta \) and \( \epsilon \) are not common knowledge, but short-lived investors can learn about them by conditioning on the market-clearing price and trading volume for the long-dated bond at time 1. In particular, given the price \( P_1 \) and volume \( 1 - \theta_{L,1} \), the short-horizon investors can compute a summary statistic \( z = (\delta + \epsilon) \delta \) from (7). The statistic \( z \) can, in turn, be used to compute an implied value of the transitory shock \( \epsilon \) given the realized price/volume pair \( P_1 \) and \( 1 - \theta_{L,1} \) together with a conjecture about \( \delta \)

\[
\epsilon = \frac{z}{\delta} - \delta.
\]

Conditional on \( z \), the short-term investor’s posterior probability that the long-term investor is the impatient type \( \delta_I \) is

\[
\pi_I^*(z) = \frac{\pi_I f_I \left( \frac{z}{\delta_I} - \delta_I \right)}{\pi_I f_I \left( \frac{z}{\delta_I} - \delta_I \right) + \pi_P f_P \left( \frac{z}{\delta_P} - \delta_P \right)}
\]

and the posterior probability of the patient type \( \delta_P \) is

\[
\pi_P^*(z) = 1 - \pi_I^*(z) = \frac{\pi_P f_P \left( \frac{z}{\delta_P} - \delta_P \right)}{\pi_I f_I \left( \frac{z}{\delta_I} - \delta_I \right) + \pi_P f_P \left( \frac{z}{\delta_P} - \delta_P \right)}.
\]

The preferences \( \delta \) and \( \epsilon \) are fully revealed at date 1 if the implied shock from (9) is not in the support of one of the densities \( f_I \) or \( f_P \). For example, if \( \frac{z}{\delta_I} - \delta_I \) is not in the support of \( f_I \) (i.e., so that \( f_I \left( \frac{z}{\delta_I} - \delta_I \right) = 0 \)), then this reveals that \( \delta = \delta_P \) and \( \epsilon = \frac{z}{\delta_P} - \delta_P \). Similarly, if \( \frac{z}{\delta_P} - \delta_P \) is not in the support of \( f_P \), then this reveals that \( \delta = \delta_I \) and \( \epsilon = \frac{z}{\delta_I} - \delta_I \). However, if, given a realization \( z \), there are two positive-probability shocks \( \epsilon_P \) and \( \epsilon_I \) such that \((\delta_P + \epsilon_P)\delta_P = (\delta_I + \epsilon_I)\delta_I = z\), then the short-term investor cannot infer \( \delta \) with certainty. In this case the equilibrium outcome
will only be partially revealing.

A useful distinction here is the difference between local preferences and their global preferences. Market prices and volumes only reveal information about those aspects of investor preferences that are relevant for the marginal demand and supply of securities under current market conditions. In general, they may not reveal enough information to infer investor preferences under all possible future conditions. For example, in this context $z = (\delta + \epsilon) \delta$ summarizes the long-lived investors’ local preferences as reflected in the date 1 bond price/volume. It is not always possible, however, to use market data to identify separately the parameters $\delta$ and $\epsilon$ that determine the long-lived investors’ global preferences.

### 2.1 Equilibrium

An equilibrium is a collection \( \{\theta_{L,1}, \theta_{L,2}, \theta_{S,1}, \theta_{S,2}, P_1, P_2\} \) of functions of the underlying state variables. At each date and in each state \((\delta, \epsilon)\), prices clear the market for the bond, the portfolio positions $\theta_{L,1}$ and $\theta_{L,2}$ maximize the long-lived investors’ utility, and $\theta_{S,1}$ maximizes the short-lived investors’ expected utility given that they use Bayes’ rule to condition on market prices and volumes at date 1 to learn about $\delta$ and $\epsilon$. We first describe equilibrium in states that are fully revealing and then the equilibrium in states that are only partially revealing.

**Fully revealing states.** In states in which the outcome is fully revealing, short-term investors infer the long-lived investors’ preferences $\delta$ and $\epsilon$ — and, hence, the future bond price $P_2$ — with certainty. Under these conditions, the short-term investor optimization problem is to choose a bond holding $\theta_{S,1}$ to smooth consumption across time:

\[
\max_{\theta_{S,1} \geq 0} u(e_{S,1} - P_1 \theta_{S,1}) + \beta u(P_2 \theta_{S,1} + e_{S,2}).
\]  

(12)

The first-order condition to this problem gives

\[
\frac{\beta u(e_{S,1} - P_1 \theta_{S,1})}{u_c(e_{S,1} - P_1 \theta_{S,1})} = \frac{P_1}{P_2}.
\]  

(13)

Using the implicit function theorem, the long-lived investor’s first-order condition (6) can be rewritten to express $P_2$ as a function of the net trade $1 - \theta_{L,1}$. The same can be done using (5).
to express $P_1$ as a function of $1 - \theta_{L,1}$ except when the long-lived investors have log preferences (i.e., relative risk aversion of 1).\footnote{We solve explicitly for the equilibrium with log preferences below.} Substituting these functions into (13) along with market clearing, $\theta_{S,1} = 1 - \theta_{L,1}$, gives an equation in $\theta_{L,1}$. Substituting the solution $\theta_{L,1}$ into (5) and (6) gives the corresponding equilibrium prices $P_1$ and $P_2$.

Since $P_2$ is riskless with full revelation, the bond is riskless. Given the Inada conditions, short-lived investors are always buying bonds given that the long-lived investors’ initial endowment $e_{L,1} = 0$.

**Partially revealing states.** When the price and volume do not fully reveal the type of the long-lived investor at date 1, the short-term investor faces liquidity risk about the resale price $P_2$ when liquidating any bond position at date 2. Given our assumptions about the long-lived investor’s preferences, two bond prices are possible at date 2 corresponding to the two possible types $\delta_I$ and $\delta_P$. Since the short-lived investor is a price-taker, she views these two bond prices as exogenous. Not knowing her counterparty’s type makes the long-term bond risky for her.

The short-term investor’s optimization problem is:

$$\max_c u(e_{S,1} - P_1\theta_{S,1}) + \beta [\pi^*_I(z)u(\theta_{S,1}P_{2,I} + e_{S,2}) + \pi^*_P(z)u(\theta_{S,1}P_{2,P} + e_{S,2})],$$

where $P_{2,I}$ is the second-period bond price if the long-term investor is impatient and $P_{2,P}$ is the second-period bond price if the long-term investor is patient.

The equilibrium prices and allocations can be recovered by imposing good market clearing,

$$c_{S,1} + c_{L,1} = e_{S,1},$$
$$c_{S,I} + c_{L,I} = e_{L,2} + e_{S,2},$$
$$c_{S,P} + c_{L,P} = e_{L,2} + e_{S,2}$$

where the consumption profile $\{c_{L,1}, c_{L,I}, c_{L,P}\}$ for the long-term investor is given in (4).
2.2 Logarithmic Preferences

When investors have logarithmic preferences $\ln(c)$, the equilibrium prices and quantities are unique and can be computed in closed-form. From (7) and (8), the long-lived investor’s net sale at date 1 and the long-dated bond’s price at date 2 are

$$1 - \theta_{L,1} = \frac{1}{(\delta + \epsilon)\delta}, \quad (16)$$

$$P_2(\delta) = \frac{(\delta + \epsilon)\delta e_{L,2}}{1 + \delta + \epsilon}, \quad (17)$$

The corresponding consumption profile of the long-term investor is

$$c_{L,1} = \frac{P_1}{(\delta + \epsilon)\delta}, \quad (18)$$

$$c_{L}(\delta) = e_{L,2} \left( \frac{\delta + \epsilon}{1 + \delta + \epsilon} \right). \quad (19)$$

Previously we introduced the notation $b_1$ as the short-lived investors’ shadow price for a one-period bill. The price of the bill is used to compute a shadow risk-premium for liquidity risk.

In the fully revealing equilibrium, the bill and bond prices are

$$b_1 = \frac{\beta e_{S,1}}{e_{S,2} + (1 + \beta)\left( \frac{e_{L,2}}{1 + \delta + \epsilon} \right)}, \quad (20)$$

$$P_1 = \frac{\beta e_{S,1}}{e_{S,2} + (1 + \beta)\left( \frac{e_{L,2}}{1 + \delta + \epsilon} \right) \frac{(\delta + \epsilon)\delta e_{L,2}}{1 + \delta + \epsilon}}.$$  

In the non-fully revealing equilibrium, the bill and bond prices are

$$b_1 = \frac{e_{S,1}\left( \frac{\pi_{P}(z)}{e_{S,2}^{\frac{e_{S,2}}{1 + \delta + \epsilon}}} + \frac{\pi_{I}(z)}{e_{S,2}^{\frac{e_{S,2}}{1 + \delta + \epsilon}}} \right)}{1 + \beta - \frac{\pi_{P}(z)}{e_{S,2}^{\frac{e_{S,2}}{1 + \delta + \epsilon}}} + \frac{\pi_{I}(z)}{e_{S,2}^{\frac{e_{S,2}}{1 + \delta + \epsilon}}}}. \quad (21)$$

and

$$P_1 = \frac{e_{S,1}e_{L,2}(\delta + \epsilon)\delta\left( \frac{\pi_{P}(z)}{e_{S,2}(1 + \delta + \epsilon) + e_{L,2}} + \frac{\pi_{I}(z)}{e_{S,2}(1 + \delta + \epsilon) + e_{L,2}} \right)}{1 + \beta - \frac{\pi_{P}(z)}{e_{S,2}^{\frac{e_{S,2}}{1 + \delta + \epsilon}}} + \frac{\pi_{I}(z)}{e_{S,2}^{\frac{e_{S,2}}{1 + \delta + \epsilon}}}}. \quad (22)$$
2.3 Numerical examples.

We compute several numerical examples to illustrate the workings of the model more concretely. Some of the intuitions from these examples are formalized in Section 2.4.

In our first example, $\pi_I = \pi_P = 0.5$ so that the patient and impatient long-lived investors are equally likely. Patient investors have a permanent discount factor $\delta_P = 2$ and their transitory time preference shock $\epsilon_P$ is uniformly distributed between -0.7875 and 0. For the impatient investors, $\delta_I = 1.5$ and $\epsilon_I$ is uniform between 0 and 1.5. The short-lived investors’ time preference is $\beta = 1.2$. The fact that some of the discount factors are larger than 1 just means that the relative value of later consumption to early consumption is high. At time 1, the short-horizon investors have an initial endowment of $e_{S,1} = 0.5$. At time 2, the long-lived investors’ unsecuritized endowment is $e_{L,2} = 0.7$ and the short-horizon investors’ second period endowment $e_{S,2} = 0.125$.

Given overlapping but non-nested supports for statistic $z$, there is a minimum value $z' = (\delta_P + \min\{\epsilon_P\})\delta_P$ and a maximum value $z'' = (\delta_I + \max\{\epsilon_I\})\delta_I$ between which the equilibrium is only partially revealing. Any realization $z$ in between $z'$ and $z''$ is possible given either the patient or the impatient type. Outside of this range the equilibrium outcome is fully revealing. A realization $z < z'$ is only possible for the impatient type $\delta_I$ (i.e., the probability of an $e_P$ such that $(\delta_P + \epsilon_P)\delta_P = z < z'$ is zero). Similarly, a $z > z''$ is only possible for the patient type $\delta_P$ (i.e., the probability of an $e_I$ such that $(\delta_I + \epsilon_I)\delta_I = z > z''$ is zero). In this example, the probability of a non-fully revealing equilibrium outcome is 80 percent.

With log preferences, it is more intuitive to work with trading volume $1 - \theta_{L,1} = 1/z$ rather than directly with $z$. Indeed, the model has a natural market microstructure interpretation. The inelastic supply of bonds from the long-lived investors at time 1 is a random order flow and the date 1 bond price is simply the point on the short-lived traders’ bond demand schedule where they are willing to absorb $1/z$ bonds. Thus, we can view the function $P_2(\theta_{L,1})$ as the liquidity supply schedule for bonds at time 1.

Figures 2a and 2b is an example of a state where the equilibrium is fully revealing. Figure 2a shows supply and demand at time 1. The solid downward sloping curve is the short-term investors’ demand if the short-term investors know they are trading against a patient long-lived investor. The dash-dot-dot (- · ·) curve is the short-term investor’s demand if they know they are trading against
an impatient long-lived investor. The dashed (---) curve in between is the short-term investors demand if they believe they are trading with a patient long-lived investor with probability 0.5 and an impatient long-lived investor with probability 0.5. In this example, the long-lived investor happens to be patient with a high transitory shock. The solid vertical line is the supply curve of a patient long-term investor with a sufficiently large transitory shock. Her supply curve is vertical due to logarithmic utility. The dashed vertical line at $1/z''$ is the supply curve of an impatient long-term investor given the highest possible transitory shock. Hence, the short-lived investor knows that there is no possible transitory shock that would cause an impatient long-lived investor to sell so few bonds. Hence, the realized trading volume at time 1 fully reveals the long-term investors type.

Figure 2b illustrates supply and demand at time 2. Here, the short-term investor inelastically sells all of her bonds bought at time 1. Hence her supply curve is vertical. The two downward sloping curves are the long-lived investors’ demands at time 2 conditional on being either patient or impatient. In this example, the upper curve is the relevant demand because the long-lived investor is patient.

Figures 3a and 3b is an example where the equilibrium is partially revealing. Figure 3a again shows supply and demand at time 1. Now the realized supply (i.e. the solid vertical line) could be the supply curve for either (a) a patient long-term investor with a small transitory shock or (b) an impatient long-lived investor with a sufficiently large transitory shock. Consequently, the short-lived investor cannot tell if she is trading against a patient or an impatient investor. The dashed (- - -) line is the demand curve given her uncertainty about the long-lived investor’s type.

In figure 3b, the short-term investor inelastically sells all of her bonds from time 1. However, since the time 1 volume did not fully reveal the long-lived investor’s type, there are two possible second period equilibrium prices corresponding to each possible investor type.

Figure 4a plots the shadow bond price $P_1$ at date 1 and the expected bond price $E_{S,1}[P_2]$ at date 2 versus trading volume $1/z$ at date 1. A small volume of sell orders $1/z$ fully reveals that the long-lived investors are the patient type. In this case, the short-lived investors know that the future resale price $P_2$ will be $\frac{\delta P_{uc}(1)}{uc(\epsilon_L,2-P_2(1-\theta_{L,1})]}$ and, hence, are willing to pay a high price $P_1$ for the bond at time 1. The prices $P_1$ and $P_2$ are decreasing in trading volume because greater selling
at date 1 means that the long-lived investors will have to buy back more bonds at date 2 which depresses the repurchase price. Since $\theta_{L,1}$ is publicly observable at time 1, this “market overhang” effect at date 2 is fully anticipated when the bond is priced on date 1.

At the critical volume $1/z''$, the price $P_1$ and the expected price $E_{S,1}[P_2]$ both have discrete jumps downward. This is because a sufficiently large realized volume $1/z$ to the right of $1/z''$ is consistent with either $\delta_P$ and $\delta_I$. Thus, now there is liquidity risk. At a second critical volume level $1/z'$ prices and expected prices discretely jump down a second time. Very large volumes $1/z$ to the right of $1/z'$ fully reveal that the long-lived investors are the impatient type. Thus, liquidity risk is non-monotone in trading volume. Liquidity risk is zero for large and small volumes and non-zero for intermediate volumes. This same non-monotonicity can also be seen in Figure 4b where the standard deviation of bond returns at date 2 is roughly 10 percent in non-fully revealing states.

Figure 4c contrasts the shadow riskless return on one period bills with the expected return on two-period bonds. Again, larger volumes correspond to greater impatience. The two returns are equal when the equilibrium is fully revealing, but there is a small but non-trivial liquidity premium in the long-dated bond’s expected return in states in which the equilibrium is only partially revealing. The liquidity premium for the long-dated bond in Figure 4d is roughly 60 basis points.

The return/volume pattern in Figure 4c is interesting for other reasons as well. Low volumes $1/z$ at the far left fully reveal that the long-lived investors are patient. As the amount of bonds the long-lived investors sell increases, short-lived investors forgo more date 1 consumption and have more date 2 consumption. This raises both the one-period interest rate and the return on the long-dated bond. At the critical volume $1/z''$, the date 1 consumption of the short-lived investors jumps up — since they now pay less for essentially the same number of bonds due to the possibility that the long-lived investor is impatient — and their expected date 2 consumption falls — since impatient investors will, on average, pay less to buy back the bonds — thereby causing both the market-clearing one-period spot rate and the expected bond return to fall. A similar story applies at $1/z''$ where it is now fully revealed that the long-lived investors are impatient. Thus, larger volumes increase the probability of impatient investors relative to the probability of patient investors and are bad news for the short-lived investors.

These price, return and volume relations provide a natural explanation for the so-called “flight
to quality” following the 1998 Russian debt default. Consider the market for non-Russian long-dated debt. Short-term investors — investors currently not holding long-dated bonds — know that long-term investors with large bond positions (such as pensions, banks, hedge funds) will react to major bond defaults. In particular, they know that long-lived investors may reduce their holding of long-dated debt in the near term for two possible reasons. One possibility is that long-lived investors will become fundamentally more bearish about long-dated debt—that is, their permanent time preference has become impatient (in our terminology) leading to low future demand for bonds. Another possibility is that long-lived investors will stay fundamentally bullish — their permanent time preference is patient — and their future bond demand will bounce back in the future. There are also transitory shocks to the long-term investors demands for bonds. If the observed sell demand for long-dated bonds is low, this fully reveals that fundamentally the long-term investors will be bullish after the current crisis passes. However if the current sell demand is too large, then short-term investors cannot tell whether the sell demand is coming from bullish investors with large short-term liquidity needs or bearish investors with small short-term liquidity needs. Consequently, prices of long-dated bonds fall due to the lower expected future resell price at time 2. Short-term bond prices rise due to the shift in consumption between current and future consumption for short-lived investors. Spreads between short-term treasuries and long-dated credit sensitive debt rises due to the liquidity risk at time 2 given the uncertainty about the future preferences of long-term investors.

Of course, if the current sell demand is sufficiently strong, then short-term investors know that the long-term bond market has “hit bottom” in that it is fully revealed that long-term investors have become fundamentally bearish. In this case, bond prices again fall due to the lower expected resell price, shadow short-term bill prices rise, but credit spreads stay normal since there is no liquidity risk premium.

Figures 5a through 5d contrast the first parameterization with an alternate parameterization in which the short-lived investor’s initial second-period endowment is doubled from 0.125 to 0.25 and the long-lived investor’s endowment $e_{L,2}$ is reduced to keep the aggregate endowment unchanged at time 2. Reducing the resources available to the long-dated investor to repurchase the bond at date 2 lowers the equilibrium prices $P_2$ at time 2 and, hence, $P_1$ at time 1. This change also
lowers the corresponding liquidity premium. The explanation for the smaller risk premium is that consumption financed by reselling the long-dated bonds at date 2 is now a smaller part of the short-term investors’ total consumption at date 2.

2.4 General Properties

Uniformly distributed transitory shocks and induce a dramatic non-monotone relation between market volume and the standard deviation and return premium for liquidity risk. The qualitative pattern generalizes to other probability distributions.

Proposition 1 Assume that investors have log preferences. If the densities $f_I$ and $f_P$ are such that

- The probability $\pi^*_I(z) = 1$ when $z$ is above a positive-probability threshold $\hat{z}$ (i.e., when the volume $1/z$ is “small”),
- The probability $\pi^*_P(z) = 1$ when $z$ is below a positive-probability threshold $\check{z}$ (i.e., when the volume $1/z$ is “large”), and
- The probabilities $\pi^*_P(z)$ and $\pi^*_I(z)$ are both non-zero for intermediate realizations of $z$,

then both the standard deviation of bond returns and the liquidity risk premium are non-monotone in $z$. In addition, discontinuities in the densities $f_I$ and $f_P$ lead to discrete jumps in prices, spreads, volatility, and risk premia relative to volume.

The price/volume relation at date 1 will again exhibit price support levels if there are regions where small increases in volume $1/z$ (i.e., decreases in $z$) cause the likelihood ratio $\pi^*_P(z)/\pi^*_I(z)$ to decrease dramatically. The discontinuous starting and stopping of the uniform density is simply an extreme example of this.

Liquidity risk prevents investors from smoothing their consumption perfectly. Thus, it is natural to wonder whether the long-lived investors can improve their welfare by simply announcing their type before the first round of trade.

Proposition 2 If the long-lived investor is patient, then her utility would be higher if her type could be credibly pre-announced. If the long-term investor is impatient, then her utility would be lower if her type were pre-announced.
Unverifiable statements by long-lived investors about their preferences are not credible. Impatient investors benefit from being confused with patient investors at date 1 because date 1 bond prices are higher — due to their higher date 2 resell value with patient investors — than if it were common knowledge that they are impatient. As a consequence, long-lived investors cannot remove liquidity risk by simply announcing their type, unless the announcements are verifiable.

3 Conclusion

Traded securities have value both due to their claim on future cash and because of the anticipated liquidity with which they can be traded in the future. If the preferences and, hence, the future demands of potential counter-parties in future trades are uncertain, then long-dated securities expose investors to liquidity risk due to their random future resale prices. In addition, a risk premium is required to clear the market for the long-dated securities with liquidity risk.

Here, we have provided a particularly simple example of liquidity risk because of preference uncertainty. In some states, short-term investors can fully infer the future demands of their counter-parties from market clearing prices and quantities. In other states, short-term investors cannot infer future demands. Interpreting the short-term investors in our model as liquidity suppliers, our results show that in some states liquidity suppliers may require a risk-premium to compensate them for liquidity risk.

The only stochastic shocks in our model are to the preferences of the long-term traders — the liquidity demanders. The only risk that liquidity suppliers face is liquidity risk. In current work we are extending our model to allow for both cash flow risk as well as liquidity risk. How much of an additional risk premium would liquidity suppliers require to compensate them for liquidity risk in the presence of cash flow risk? How large is the liquidity risk premium relative to the cash flow risk premium? How is trading volume related to the liquidity risk premium?
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APPENDIX

Proof of Proposition 1. By hypothesis there is no uncertainty about whether the long-lived investors are patient or impatient when \( z \) is either sufficiently large or small. Thus, for extreme small and large volumes the bond is riskless at date 2. Hence, both the volatility of \( P_2 \) and the liquidity risk premium are zero. However there is uncertainty about the type of the long-lived investors for intermediate values of \( z \) and, hence, positive price volatility and a positive liquidity premium.

Proof of Proposition 2. The long-term investor’s consumption profile in either the fully revealing or partially revealing equilibrium is

\[
\begin{align*}
c_{L,1} &= \frac{P_1}{(\delta + \epsilon)\delta}, \\
c_L(\delta) &= e_{L,2} \left( \frac{\delta + \epsilon}{1 + \delta + \epsilon} \right).
\end{align*}
\]

The investor’s date 2 consumption is invariant to whether her type is revealed at date 1. However, her date 1 consumption is increasing in the date 1 price of the bond \( P_1 \).

To prove the proposition, it is sufficient to show \( P_{1,I}^{FR} < P_1 < P_{1,P}^{FR} \) where \( P_{1,I}^{FR} \) and \( P_{1,P}^{FR} \) denote the fully revealing prices of the bond conditional on \( I \) or \( P \). \( P_1 \) denotes the partially revealing price. We show \( P_{1,I}^{FR} < P_1 \). The other inequality follows immediately.

The prices \( P_{1,I}^{FR} \) and \( P_1 \) are given by

\[
\begin{align*}
P_{1,I}^{FR} &= \frac{\beta e_{S,1}}{e_{S,2} + (1 + \beta) \left( \frac{e_{L,2}}{1 + \delta_I + \epsilon_I} \right)} \frac{(\delta_I + \epsilon_I)\delta_I e_{L,2}}{1 + \delta_I + \epsilon_I} = \frac{\beta e_{S,1} e_{L,2} (\delta_I + \epsilon_I)\delta_I}{e_{S,2}(1 + \delta_I + \epsilon_I) + (1 + \beta)e_{L,2}}, \\
P_1 &= \frac{e_{S,1} e_{L,2}(\delta + \epsilon)\delta \left( \frac{\pi_{p}^{(z)}}{e_{S,2}(1 + \delta_P + \epsilon_P) + e_{L,2}} + \frac{\pi_{I}^{(z)}}{e_{S,2}(1 + \delta_I + \epsilon_I) + e_{L,2}} \right)}{1 + \frac{1 + \beta}{\beta} - e_{S,2} \left( \frac{\pi_{P}^{(z)}}{e_{S,2} + \tau \pi_{p}^{(z)}/\tau p} + \frac{\pi_{I}^{(z)}}{e_{S,2} + \tau \pi_{I}^{(z)}/\tau I} \right)}.
\end{align*}
\]
Computing $P_1 - P_{1,I}^{FR} > 0$,

$$P_1 - P_{1,I}^{FR} = \frac{e_{S,1} e_{L,2} (\delta + \epsilon) \delta \left( \frac{\pi_p^*(z)}{e_{S,2}(1+\delta_P+\epsilon_P)+e_{L,2}} + \frac{\pi_I^*(z)}{e_{S,2}(1+\delta_I+\epsilon_I)+e_{L,2}} \right) - \frac{1+\beta}{\beta} e_{S,2} \left( \frac{\pi_p^*(z)}{e_{S,2}+1+\delta_P+\epsilon_P} + \frac{\pi_I^*(z)}{e_{S,2}+1+\delta_I+\epsilon_I} \right) - \frac{\epsilon}{e_{S,2}(1+\delta_I+\epsilon_I) + (1+\beta)e_{L,2}}}{\beta e_{S,1} e_{L,2} (\delta + \epsilon) \delta}.$$

Since $(\delta + \epsilon) \delta = (\delta_I + \epsilon_I) \delta_I$ in the partially revealing equilibrium,

$$\frac{1}{\beta e_{S,1} e_{L,2} (\delta + \epsilon) \delta} (P_1 - P_{1,I}^{FR}) = \frac{\pi_p^*(z)}{e_{S,2}(1+\delta_P+\epsilon_P)+e_{L,2}} + \frac{\pi_I^*(z)}{e_{S,2}(1+\delta_I+\epsilon_I)+e_{L,2}} - \frac{1}{1+\beta - \beta e_{S,2} \left( \frac{\pi_p^*(z)}{e_{S,2}+1+\delta_P+\epsilon_P} + \frac{\pi_I^*(z)}{e_{S,2}+1+\delta_I+\epsilon_I} \right)}.$$

Rearranging the first term on the right side of (23),

$$\frac{\pi_p^*(z)}{e_{S,2}(1+\delta_P+\epsilon_P)+e_{L,2}} + \frac{\pi_I^*(z)}{e_{S,2}(1+\delta_I+\epsilon_I)+e_{L,2}}$$

$$= \frac{\pi_p^*(z)(e_{S,2}(1+\delta_I+\epsilon_I)+e_{L,2}) + \pi_I^*(z)(e_{S,2}(1+\delta_P+\epsilon_P)+e_{L,2})}{e_{S,2}(1+\delta_P+\epsilon_P)+e_{L,2})} - \frac{1}{1+\beta - \beta e_{S,2}(\pi_p^*(z)(1+\delta_P+\epsilon_P) + \pi_I^*(z)(1+\delta_I+\epsilon_I))}$$

$$= \frac{\pi_p^*(z) m_I + \pi_I^*(z) m_p}{(1+\beta) m_p m_I - \beta (\pi_p^*(z)(m_p - e_{L,2}) m_I + \pi_I^*(z)(m_I - e_{L,2}) m_p)}$$

where $m_I \triangleq e_{S,2}(1+\delta_I+\epsilon_I) + e_{L,2}$ and $m_p \triangleq e_{S,2}(1+\delta_P+\epsilon_P) + e_{L,2}$. Note that $m_I > m_p$ when $\delta_P > \delta_I$.

Substituting back into (23),

$$\frac{1}{\beta e_{S,1} e_{L,2} (\delta + \epsilon) \delta} (P_1 - P_{1,I}^{FR}) = \frac{\pi_p^*(z) m_I + \pi_I^*(z) m_p}{(1+\beta) m_p m_I - \beta (\pi_p^*(z)(m_p - e_{L,2}) m_I + \pi_I^*(z)(m_I - e_{L,2}) m_p)} - \frac{1}{m_I + \beta e_{L,2}}.$$

Computing a common denominator on the right side of (24), its numerator determines the sign
of $P_1 - P_{1,J}^{FR}$ and is

$$(m_I + \beta e_L,2)(\pi_p^*(z)m_I + \pi_I^*(z)m_p)$$

$$-(1 + \beta)m_pm_I + \beta(\pi_p^*(z)(m_p - e_L,2)m_I + \pi_I^*(z)(m_I - e_L,2)m_p)$$

$$= m_I(\pi_p^*(z)m_I + \pi_I^*(z)m_p) - m_pm_I > 0$$

since $\pi_p^*(z)m_I + \pi_I^*(z)m_p > m_p$. This establishes $P_1 - P_{1,J}^{FR} > 0$.  ■
Long-term investor learns \((\delta, \epsilon)\)

First round of trade: \(P_1, \theta_{L,1}\)  
First period consumption

Second round of trade: \(P_2\)
Second period consumption

Third period consumption

Bond pays off

Figure 1: The Time-Line
Figure 2a: **Example of a Fully Revealing Equilibrium — First Period**

Figure 2b: **Example of a Fully Revealing Equilibrium — Second Period**
Figure 3a: Example of a Partially Revealing Equilibrium — First Period.

Figure 3b: Example of a Partially Revealing Equilibrium — Second Period
Figure 4a: **Long Bond Prices Versus Trade.** The dashed line plots the expected $t = 2$ long bond price versus trade in the long bond. The solid line is the $t = 1$ long bond price versus trade in the long bond. Parameters: $e_{S,1} = 0.5$, $\alpha_{S,2} = 0.125$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, Prob(Not Fully Revealing) = 0.8.

Figure 4b: **Long Bond Return Volatility Versus Trade.** Parameters: $e_{S,1} = 0.5$, $\alpha_{S,2} = 0.125$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, Prob(Not Fully Revealing) = 0.8.
Figure 4c: **Bond Rates of Return Versus Trade.** The dashed line is the expected rate of return of the long bond versus trade in the long bond. The solid line is the rate of return of the bill versus trade in the long bond. Parameters: $e_{S,1} = 0.5$, $\alpha_{S,2} = 0.125$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, $\text{Prob(Not Fully Revealing)} = 0.8$.

Figure 4d: **Liquidity Premium Versus Trade.** The liquidity premium is plotted across the entire feasible trade region. Parameters: $e_{S,1} = 0.5$, $\alpha_{S,2} = 0.125$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, $\text{Prob(Not Fully Revealing)} = 0.8$.  

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Figure 5a: **Long Bond Prices Versus Trade.** The dashed lines plots the expected $t = 2$ long bond price versus trade in the long bond. The solid lines are the $t = 1$ long bond prices versus trade in the long bond. The thick lines represent an equilibrium where the bill endowment is doubled to $\alpha_{S,2} = 0.25$ but the aggregate endowment remains fixed. The thin lines represent an equilibrium where $\alpha_{S,2} = 0.125$. Parameters: $e_{S,1} = 0.5$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, Prob(Not Fully Revealing) = 0.8.

Figure 5b: **Long Bond Return Volatility Versus Trade.** The thick line represents an equilibrium where the bill endowment is doubled to $\alpha_{S,2} = 0.25$ but the aggregate endowment remains fixed. The thin line represents an equilibrium where $\alpha_{S,2} = 0.125$. Parameters: $e_{S,1} = 0.5$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, Prob(Not Fully Revealing) = 0.8.
Figure 5c: **Bond Rates of Return Versus Trade.** The dashed line is the expected rate of return of the long bond versus trade in the long bond. The solid line is the rate of return of the bill versus trade in the long bond. The thick lines represent an equilibrium where the bill endowment is doubled to $\alpha_{S,2} = 0.25$ but the aggregate endowment remains fixed. The thin lines represent an equilibrium where $\alpha_{S,2} = 0.125$. Parameters: $e_{S,1} = 0.5$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, $\text{Prob(Not Fully Revealing)} = 0.8$.

Figure 5d: **Liquidity Premium Versus Trade.** The liquidity premium is plotted across the entire feasible trade region. The thick line represents an equilibrium where the bill endowment is doubled to $\alpha_{S,2} = 0.25$ but the aggregate endowment remains fixed. The thin line represents an equilibrium where $\alpha_{S,2} = 0.125$. Parameters: $e_{S,1} = 0.5$, $e_{L,2} = 0.7$, $\delta_P = 2$, $\delta_I = 1.5$, $\beta = 1.2$, $\pi_P = 0.5$, $\text{Prob(Not Fully Revealing)} = 0.8$. 

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